The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme
Final Examination 2017/2018
Applied Mathematics – Level 04
ADU4302/APU 2143– Vector Calculus
Duration :- Two Hours.

Date :-22.09.2018 **Time:- 1.30 p.m. - 3.30 p.m.**

Answer Four Questions Only.

1. (a) State and sketch the domain of the function $f(x, y)=\dfrac{\sqrt{1+x+y}}{x-1}$.

(b) Sketch the level curves of the function $f(x, y)=\dfrac{1}{x^2+y^2+1}$.

(c) Find the following limits, if they exist:

(i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy+y^2}{x^2+y^2}$, (ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$.

(d) Discuss the continuity of the following functions, at the origin:

(i) $f(x, y)=\begin{cases} \dfrac{x^2-xy+y^2}{x^2+y^2} & \text{if } (x, y)\neq(0, 0) \\ 0 & \text{if } (x, y)=(0, 0). \end{cases}$

(ii) $f(x, y)=\begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} & \text{if } (x, y)\neq(0, 0) \\ 2 & \text{if } (x, y)=(0, 0). \end{cases}$

2.(a) If a scalar field in Cartesian coordinates is $f(x,y)$ and the corresponding scalar field in plane polar coordinates is $g(r,\theta)$ then $f(x,y)=g(r,\theta)$.

The relationships between the coordinates are given by $x=r\cos\theta$ and $y=r\sin\theta$.

(i) Show that $\dfrac{\partial r}{\partial x}=\cos\theta$ and $\dfrac{\partial r}{\partial y}=\sin\theta$

(ii) Show that $\dfrac{\partial \theta}{\partial x}=-\dfrac{\sin\theta}{r}$ and $\dfrac{\partial \theta}{\partial y}=\dfrac{\cos\theta}{r}$

(iii) Hence show that $f_x=g_r\cos\theta-g_\theta\sin\theta/r$ and $f_y=g_r\sin\theta+g_\theta\cos\theta/r$.

Here f_x, f_y have a standard meaning. Similarly for the others.

(iv) Show that $f_{xx} = g_{rr}\cos^2\theta - g_{r\theta}\dfrac{2\sin\theta\cos\theta}{r} + g_\theta\dfrac{2\sin\theta\cos\theta}{r^2} + g_r\dfrac{\sin^2\theta}{r} + g_{\theta\theta}\dfrac{\sin^2\theta}{r^2}$ and find a similar expression for f_{yy}.

(v) Hence determine a formula in polar coordinates for $\nabla^2 f = f_{xx} + f_{yy}$.

(b) (i) If $g = r^2\sin 2\theta$ then find $f(x,y)$ and $\nabla f(x,y)$ in x and y.

(ii) Find the directional derivative at the point (1, 3) in the direction parallel to (3, 4).

3. (a)(i) Prove that grad ϕ is a vector normal to the contour surface $\phi(x,y,z) = c$, where c is a constant.

(ii) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by $(x-x_0)\left(\dfrac{\partial F}{\partial x}\right)_P + (y-y_0)\left(\dfrac{\partial F}{\partial y}\right)_P + (z-z_0)\left(\dfrac{\partial F}{\partial z}\right)_P = 0$.

(iii) Using the above result, find the equation of the tangent plane to the surface $F(x, y, z) = e^x\cos y$ at the point (0, 0, 1).

(b) (i) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D. Explain briefly how you could determine the nature of the stationary point.

(ii) Find the maximum and minimum values of the function $f(x, y) = 4x^2 + 4y^2 + x^4 - 6x^2y^2 + y^4$ and determine their nature.

4. (a) State Gauss' Divergence theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = (z+a)\underline{k}$ taken over the entire surface of the solid hemisphere $x^2 + y^2 + z^2 \le a^2$ and $z \ge 0$.

(c) Let $\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ and $r = |\underline{r}|$ then prove the following.

(i) $\underline{\nabla}\cdot\underline{r} = 3$,
(ii) $\underline{\nabla}\times\underline{r} = \underline{0}$,
(iii) $\underline{\nabla}\, r^n = nr^{n-2}\underline{r}$ where n is a constant,

(iv) $\underline{\nabla}\cdot r^n \underline{r} = (n+3)r^n$, (Hint $\underline{\nabla}\cdot(f\underline{A}) = \underline{\nabla} f\cdot\underline{A} + f\underline{\nabla}\cdot\underline{A}$),

(v) $\underline{\nabla}\times(r^n\underline{r}) = \underline{0}$, (Hint $\underline{\nabla}\times(f\underline{A}) = \underline{\nabla} f\times\underline{A} + f\underline{\nabla}\times\underline{A}$).

5. (a) State Stokes' Theorem.

(b) Verify Stokes' Theorem considering the vector field $\underline{F} = xz\underline{j}$ defined over the section of a sphere of radius a and $0\le\theta\le\alpha$.

(c) Prove that the vector field $\underline{F} = (e^x(x\cos y+\cos y - y\sin y),\ e^x(-x\sin y - \sin y - y\cos y),\ 0)$ is irrotational and find a corresponding scalar potential function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

6. (a) Suppose that S is a plane surface lying in the xy-plane, bounded by a closed curve C. If $\underline{F} = P(x,y)\underline{i} + Q(x,y)\underline{j}$ then show that $\oint_C (P\,dx + Q\,dy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$.

(b) Verify the above result for the integral $\oint_C x^2y\,dx + xy^2\,dy$, where C is the closed curve formed by $y^2 = 4ax$ and $x = a$; $a > 0$.