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The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2017/2018
Pure Mathematics - Level 04
PEU4300/PEE4300– Real Analysis I

Duration: - Two Hours

Date: - 17-09-2018 **Time: -1.30 p.m –3.30 p.m.**

Answer **Four** questions only.

1 (a) Prove that the sequence $\left\langle\frac{10n-1}{50-n^2}\right\rangle$ is bounded. Write down the greatest lower bound and the least upper bound. Find whether the sequence is monotonic. Justify your answer.

(b) Let $a_n = n - 1 - \sqrt{n^2 - 1}$. Using the definition of convergent sequence, prove that the sequence $\langle a_n \rangle$ is convergent.

(c) Suppose the sequence $\langle b_n \rangle$ is convergent and the set $\{b_n : n \in \mathbb{N}\}$ is finite. Prove that there exists $n_0 \in \mathbb{N}$ such that $b_n = c$ for all $n \geq n_0$, where c is a real number.

2. Let a and D be positive real numbers. The sequence $\langle s_n \rangle$ is defined recursively by,
$s_1 = a\,,\quad s_{n+1} = \frac{1}{2}\left(s_n + \frac{D}{s_n}\right)$ for $n \in \mathbb{N}$.

(i) Prove that $s_{n+1} \geq \sqrt{D}$ for $n \geq 2$.

(ii) Prove that $s_{n+1} \leq s_n$ for $n \geq 2$.

(iii) Deduce that $\langle s_n \rangle$ is convergent and find its limit.

(iv) Prove that $\langle s_n \rangle$ is a constant sequence iff $s_1 = \sqrt{D}$.

3. (i) Show that the sequence $\left\langle \sum_{k=1}^{n} \frac{1}{(k-1)!} \right\rangle$ is bounded above. (Recall that $0! = 1$)

(ii) Show that the sequence $\left\langle \left(1 + \frac{1}{n}\right)^n \right\rangle$ is strictly increasing.
(Hint: consider $c_k = 1 + \frac{1}{n}$ for $k = 1, 2, 3, \ldots n$ and $c_{n+1}=1$ and use the arithmetic geometric inequality $\frac{1}{n+1}(c_1 + c_2 + \ldots + c_{n+1}) > (c_1 c_2 \ldots c_{n+1})^{\frac{1}{n+1}}$ for positive reals $c_1, c_2, \ldots c_{n+1}$.}

(iii) Show that the sequence $\langle (1 + \frac{1}{n})^n \rangle$ is bounded above.

4 (a) Prove that the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converge iff $|r| < 1$.

(b) Let r be a real number such that $0 < r < 1$ and $m \in \mathbb{N}$.
Prove that $0 < r^m < \frac{r}{m(1-r)}$.

(c) Prove that $\sum_{i=(n+1)}^{2n} \frac{1}{i} \geq \frac{1}{2}$, for any $n \in \mathbb{N}$.
Deduce that $\sum_{i=1}^{2^m} \frac{1}{i} \geq 1+\frac{m}{2}$ for any $m \in \mathbb{N}$.

5 (a) Find whether each of the following series is convergent or divergent. Justify your answer.

(i) $\sum_{n=1}^{\infty} \left(\sqrt{n+1} - \sqrt{n+2}\right)$

(ii) $\sum_{n=1}^{\infty} \frac{(n^2+10)}{(1-2n+n^2)}$

(iii) $\sum_{n=1}^{\infty} \frac{1}{(2^n+(-2)^n+2)}$

(iv) $\sum_{n=1}^{\infty} \left(\frac{2^n.\ n^{100}}{3^n}\right)$

(v) $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} r^{n+1}, & n \text{ is even} \\ r^{n-1}, & n \text{ is odd} \end{cases}$ and $0 < r < 1$.

6 (a) Find the lim sup and lim inf of each of the following sequences.

(i) $\left\langle \sin\left(\frac{n\pi}{3}\right) \right\rangle$ (ii) $\left\langle (-1)^n \left(2 + \frac{3}{n}\right) \right\rangle$ (iii) $\left\langle \sum_{k=1}^{n} \frac{(-1)^k}{2^k} \right\rangle$

(b) Find the radius of convergence of each of the following power series.

(i) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right) x^n$ (ii) $\sum_{n=1}^{\infty} a_n x^n$, where $a_n = \begin{cases} \frac{1}{2^n}, & n \text{ is even} \\ \frac{1}{n}, & n \text{ is odd} \end{cases}$.