The Open University of Sri Lanka

B.Sc./B.Ed. Degree Programme

Final Examination-2017/2018

ADU5302/ADE5302/APU3143-Mathematical Methods

Applied Mathematics -Level 05

Duration: Two Hours.

Date: 12.09.2018	Time: 9.30 a.m.- 11.30a.m.

Answer FOUR questions only.

1. (a) Find the Laplace transform of $t^2e^{-2t}\cos t$.

   (b) Find the inverse Laplace transform of each of the following:

   (i) $\dfrac{5s+10}{9s^2-16}$ (ii) $\dfrac{s^2}{\left(s^2-4s+5\right)^2}$

   (c) Using the convolution theorem, find the inverse Laplace transform of $\dfrac{1}{(s+1)(s^2+1)}$.

   (d) Solve the following boundary value problem using the Laplace transform method:

   $$4\frac{d^2y}{dx^2}-4\frac{dy}{dx}-8y = 5e^{4x}, \text{ subject to } y(0)=2, \quad y'(0)=1$$

2. (a) Define the orthogonal system and the orthonormal system of functions.

   (b) Consider the boundary value problem

   $$\frac{d^2y}{dx^2}+\mu y=0$$
   $$y(-2)=y(2)$$
   $$y'(-2)=y'(2)$$

   (i) Show that this is a Sturm-Liouville problem.

   (ii) Find the eigenvalues and eigenfunctions of the problem.

   (iii) Obtain a set of functions, which are orthonormal in the interval $-2\le x\le 2$.

3. (a) Find the Fourier Series of $f(x) = x^2 - 2$; $-2 < x < 2$

(b) Find the half range cosine series if $f(x) = \begin{cases} \pi x & ;0 < x < 1 \\ \pi(2-x) & ;1 < x < 2 \end{cases}$

(c) Find the half range sine series if $f(x) = e^x$;$0 \le x \le \pi$

4. (a) The Gamma function denoted by $\Gamma(p)$ corresponding to the parameter p is defined by

the improper integral $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt, \quad (p > 0).$

Compute each of the following:

(i) $\frac{\Gamma(5)\Gamma(4.5)}{\Gamma(2.5)}$. (ii) $\Gamma(-4.5)$ (iii) $\int_0^\infty \sqrt{x} e^{-\sqrt[3]{x}} dx$.

(b) The Beta function denoted by $\beta(p,q)$ is defined by $\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx,$

where are positive parameters.

Prove each of the following results using Beta function:

(i) $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}}$; where $m, n > 0$

Hence show that $\int_0^1 (x \log x)^4 dx = \frac{4!}{5^5}$.

(ii) $\int_0^1 \sqrt{x}\, \sqrt[3]{1-x^2}\, dx = \frac{\Gamma(3/4)\Gamma(4/3)}{2\Gamma(7/12)}$.

5. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p}.m!\Gamma(p+m+1)}.$$

(a) Show that $\frac{d}{dx}\left[J_n^2 + J_{n+1}^2\right] = 2\left[\frac{n}{x} J_n^2 - \frac{n+1}{x} J_{n+1}^2\right]$.

(b) Evaluate $\int x^2 J_1(x) dx$.

(c) Express $J_{\frac{5}{2}}(x)$ in terms of sine and cosine functions.

$$\left(\text{Hint}: J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x\right)$$

(d) Prove that $\int J_3(x)dx + J_2(x) + \frac{2}{x}J_1(x) = 0$

6. The Rodrigues' formula for the n^{th} degree Legendre polynomial denoted by $P_n(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

$P_n(x)$ is also given by the sum

$$P_n(x) = \sum_{m=0}^{M}\frac{(-1)^m(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}, \quad n = 0,\ 1,\ 2, \ldots,$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$ whichever is an integer.

Using this expansion prove each of the following:

(a) $xP'_n(x) = nP_n(x) + P'_{n-1}(x)$.

(b) $(1-x^2)P'_{n-1} = n(xP_{n-1} - P_n)$.

$$\left(\text{Hint}: P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x),\ P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).\right)$$

(c)(i) $P_n'(1) = \frac{n(n+1)}{2}$.

(ii) $P_n'(-1) = (-1)^n\frac{n(n+1)}{2}$.

(d) $$\int_{-1}^{1}(1-x^2)P_m'P_n'dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2n(n+1)}{2n+1} & \text{if } m = n \end{cases}$$

where m and n are positive integers.