The Open University of Sri Lanka
Department of Mathematics
B.Sc/ B.Ed Degree Programme
Final Examination - 2017/ 2018
Pure Mathematics– Level 05
PEU5304 – Introduction to Complex Analysis

DURATION: TWO HOURS

Date: 04 – 10 – 2018 | **Time: 9.30 a.m. – 11.30 a.m.**

ANSWER FOUR QUESTIONS ONLY

01. Let z be a complex number. Prove or disprove each of the following statements:

(a) There is no pure imaginary number z satisfying $z^2 = z$.

(b) If $\operatorname{Im}(z)$ non-negative, then $\operatorname{Im}(z^{-1})$ is also non-negative.

(c) $\left[(-1)^2\right]^{1/2} = \left[(-1)^{1/2}\right]^2$.

(d) $\exp(z)$ is real if and only if $\operatorname{Im}(z) = n\pi,\ n \in \mathbb{Z}$.

(e) For each $z \in \mathbb{C}$, $\cosh^2 z - \sinh^2 z = 1$.

(f) For all $z \in \mathbb{C} - \{0\}$, $A\operatorname{rg}(\bar{z}) = A\operatorname{rg}(z)$.

(g) z is a real number if and only if $z = \bar{z}$.

(h) $z\bar{z} \geq 0$ and $z\bar{z} = 0$ if and only if $z = 0$.

02. Let z_1 and z_2 be two complex numbers.

(a) Prove that $|z_1 + z_2|^2 = |z_1|^2 + 2\operatorname{Re}(z_1\bar{z}_2) + |z_2|^2$.

Hence, derive the triangle inequality for complex numbers.

Deduce that $\big||z_1| - |z_2|\big| \leq |z_1 - z_2| \leq |z_1| + |z_2|$.

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(b) Show each of the following inequalities:

(i) $\left|\dfrac{z_1+z_2}{z_3-z_4}\right| \le \dfrac{|z_1|+|z_2|}{\left||z_3|-|z_4|\right|}$ when $|z_3| \ne |z_4|$,

(ii) $\left|\mathrm{Re}\left(6-\bar{z}_1+3z_1^2\right)\right| < 20$ when $|z_1|<2$,

(iii) If z_2 lies on the circle $|z_2|=3$, then $\left|\dfrac{1}{z_2^4-5z_2^2+4}\right| \le \dfrac{1}{40}$.

03. Let z_1 and z_2 be two non-zero complex numbers.

(a) Show that $z_1\,\bar{z}_2$ is a non-zero complex number with $|z_1\,\bar{z}_2| = |z_1||z_2|$ and

$\arg(z_1\,\bar{z}_2) = \arg(z_1) - \arg(z_2)$.

Hence, write down the product of the complex number $-1+i$ and $\overline{\sqrt{3}-i}$ in polar form.

Is $\left(\dfrac{\sqrt{3}-i}{2}\right)^{2018} = \left(\dfrac{1-\sqrt{3}\,i}{2}\right)$? Justify your answer.

(b) Prove that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$.

04. (a) Locate the complex numbers $z_1 = 5+2i$, $z_2 = -1+4i$ and $z_3 = 4+6i$ in a complex plane.

Represent z_1+z_2 and $2z_2-z_3$ in the same complex plane.

(b) Find the locus of the points in a complex plane that represents each of the following complex numbers z such that:

(i) $A\mathrm{rg}(z+2-3i) = \dfrac{4\pi}{5}$,

(ii) $|(1-i)z+2| = 4$,

(iii) $\mathrm{Re}\left(\dfrac{z-5i}{z+3i}\right) = 0$ when $z \ne 3i$.

05. Let θ be any real number and let n be any integer. State De Moivre's theorem for complex numbers.

(a) Establish the identity $1+z+z^2+\ldots+z^n=\dfrac{1-z^{n+1}}{1-z}$ for $z\in\mathbb{C}-\{1\}$.

Hence, derive that $1+\cos\theta+\cos 2\theta+\ldots+\cos n\theta=\dfrac{1}{2}+\dfrac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\left(\frac{\theta}{2}\right)}$, $0<\theta<2\pi$.

(b) Show that for all z_1, $z_2\in\mathbb{C}$, if $\exp(z_1)=\exp(z_2)$ then $z_1=z_2+2n\pi i$ for some $n\in\mathbb{Z}$.

Hence, find all the complex numbers z such that $\exp(z)=1$.

06. Let z be a complex number.

(a) Find the four roots of the equation $z^4+4=0$.

Hence, use them to factor z^4+4 into quadratic factors with real coefficients.

(b) Find the primitive fourth roots of unity.

(c) Verify that $1-2i$ is a factor of the polynomial $p(z)=z^4+3z^2+6z+10$.

Find all the roots of the equation $p(z)=0$.