00109

The Open University of Sri Lanka
Department of Mathematics
B.Sc/ B.Ed Degree Programme
Final Examination - 2017/ 2018
Pure Mathematics– Level 05
PEU5302– Combinatorics

DURATION: TWO HOUR

Date: 16 – 09 – 2018 **Time: 09.30 a.m. – 11.30 a.m.**

ANSWER FOUR QUESTIONS ONLY

01. State the *binomial theorem*. Using the binomial expansion of $(1+x)^n$ where $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, prove each of the following:

(a) $c_0 - c_1 + c_3 \cdots - c_n = 0$, when n is odd.

(b) $c_0^2 - c_1^2 + c_2^2 - \cdots - c_n^2 = 0$, when n is odd.

(c) $c_1 + 2c_2 + 3c_3 \cdots + nc_n = n2^{n-1}$.

(d) $c_0 + \dfrac{c_1}{2} + \dfrac{c_2}{3} + \cdots + \dfrac{c_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$.

where $c_r \left(= {}^nC_r\right)$ is the coefficient of the $(r+1)^{\text{th}}$ term of the expansion of $(1+x)^n$.

02. (a) Find the sum of the coefficients of the terms with *even powers* of x of the multinomial expansion of $(1+x-x^2)^n$.

Hence, show that the sum of the coefficients of the terms with *odd powers* of x of the multinomial expansion is $\dfrac{1+(-1)^{n+1}}{2}$.

(b) Find the multinomial coefficient of x^7 in the expansion of $\left(p-qx+rx^2\right)^8\left(p+qx-rx^2\right)^8$.

03. Let n, m and r be three positive integers such that $1 \leq r \leq n$ and $1 \leq r \leq m$. Prove each of the following using ONLY the *combinatorial arguments*:

(a) ${}^{3n}C_2 - {}^{2n}C_2 - {}^{n}C_2 = 2n^2$, by counting the number of ways of selecting a teacher and a student from an assembly of n teachers and $2n$ students.

(b) $n \times {}^{n-1}C_{r-1} = r \times {}^{n}C_r$, by counting the number of ways of selecting a r- person committee and the committee chair from a board of n members.

(c) ${}^{n}C_r = {}^{n-2}C_r + 2\,{}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$, by counting the number of ways of selecting r members from a set of n members.

(d) ${}^{n}C_2 + {}^{m}C_2 + nm = {}^{n+m}C_2$, by counting the number of ways of selecting 2 persons from a group of n men and m women.

04. (a) Briefly explain that the number of ways of selecting n *identical* objects from t *different* groups, where each group contains at least n objects of the similar kind, is the same as the number of *integer solutions* to $x_1 + x_2 + \ldots + x_t = n$ subject to $x_i \geq 0$ for $i \in \{1, 2, \ldots, t\}$.

(b) (i) Suppose the order in which the fruits are chosen does not matter and each fruit type is identical. How many ways are there to choose 5 fruits from a fruit shop containing Apple, Orange, Pears, Pineapple, Mango, Avocado, and Guava?

(ii) How many ways are there to place 10 *identical* balls into 8 different bowls?
If these 10 balls are *different*, find the number of ways to place them in those 8 bowls.

(iii) A person has 12 ten rupee coins and he wants to distribute it among 6 beggars so that each beggar will get at least one coin.
Model this as an integer problem by clearly stating all necessary constrains.
Hence, find the number of ways of distributing those coins to the beggars.

05. Let A and B be any two events in a probability space S. Explain briefly the following:

(i) A and B are *mutually exclusive events.*

(ii) A and B are *independent events.*

(a) Without using *Venn diagram*, prove each of the following:

(i) $P(A) \le P(B)$ whenever $A \subseteq B$.

(ii) $P(A^c) = 1 - P(A)$ where A^c is the complement event of A.

(iii) $P(A \setminus B) = P(A) - P(A \cap B)$.

Hence, deduce the *addition law* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(b) Find $P(A \cup B)$ when

(i) $A \subseteq B$.

(ii) A and B are *independent events.*

(iii) A and B are *mutually exclusive events.*

(c) A number is randomly selected between 1 and 600 inclusive. What is the probability that the number is a multiple of 12 but not a multiple of 20?

06. (a) Let a and b be two real numbers. Let x_1 and x_2 be two *distinct roots* of the equation $x^2 = ax + b$. If $f_n = cx_1^n + dx_2^n$ for some real numbers c and d, then prove that f_n satisfies the *difference equation* $f_n = af_{n-1} + bf_{n-2}$.

(b) Let f_n be the number of series which contain *ones and twos only*, where $n \in \mathbb{N}$.

If sum of f_n is equal to n.

(i) Write down first 4 terms of f_n.

(ii) Construct a *difference equation* satisfied by f_n for all $n \ge 3$.

Hence, show that $f_1 + f_3 + \ldots + f_{2n-1} = f_{2n} - 1$.

If the sum of the series which contain ones and twos only is 7, find the number of all possible series.