The Open University of Sri Lanka
Department of Mathematics
B.Sc/ B.Ed Degree Programme
Final Examination - 2017/ 2018
Pure Mathematics– Level 05
PUU3242– Combinatorics

DURATION: THREE HOUR

Date: 16 – 09 – 2018 **Time: 09.30 a.m. – 12.30 p.m.**

ANSWER FIVE QUESTIONS ONLY

01. (a) Find the sum of the coefficients of the terms with *even powers* of x of the multinomial expansion of $(1+x-x^2)^n$.

Hence, show that the sum of the coefficients of the terms with *odd powers* of x of the multinomial expansion is $\dfrac{1+(-1)^{n+1}}{2}$.

(b) Find the multinomial coefficient of x^7 in the expansion of $\left(p-qx+rx^2\right)^8\left(p+qx-rx^2\right)^8$.

(c) Are the coefficients of $x^2yz^2w^5$ and $x^2y^2zw^5$ in the expansion of $\left(x-2y-z+w\right)^{10}$ the same? Justify your answer.

02. State the *binomial theorem.* Using the binomial expansion of $(1+x)^n$ where $x\in\mathbb{R}$ and $n\in\mathbb{Z}^+$, prove each of the following:

(a) $c_0+c_2+c_4\cdots+c_{n-1}=2^{n-1}$, when n is odd.

(b) $c_0-c_1+c_3\cdots-c_n=0$, when n is odd.

(c) $c_0^2-c_1^2+c_2^2-\cdots-c_n^2=0$, when n is odd.

(d) $c_1+2c_2+3c_3\cdots+nc_n=n2^{n-1}$.

(e) $c_0+\dfrac{c_1}{2}+\dfrac{c_2}{3}+\cdots+\dfrac{c_n}{n+1}=\dfrac{2^{n+1}-1}{n+1}$.

where $c_r\left(={}^nC_r\right)$ is the coefficient of the $(r+1)^{\text{th}}$ term of the expansion of $(1+x)^n$.

03. State the *generalized pigeonhole principle.*

(a) Show that from 25 distinct even positive integers all less than 100, there is a pair of numbers whose sum is 102.

(b) Show that in a group of 25 people at least 4 were born on the same day of the week.
How many people are needed to be able to assert with certainty that 3 have the same birth day in a week?

(c) Suppose that in a group of 6 people, each pair of individuals consists of 2 friends or 2 enemies. Show that there are either 3 mutual friends or 3 mutual enemies in the group.

(d) There are 25 students in the class and the sum of their marks of a subject is 1380. Is it true that one can find 15 students in the class such that sum of their marks greater than 825?

04. Let A and B be any two events in a probability space S. Explain briefly the following:

(i) A and B are *mutually exclusive events.*

(ii) A and B are *independent events.*

(a) Without using *Venn diagram*, prove each of the following:

(i) $P(A) \leq P(B)$ whenever $A \subseteq B$.

(ii) $P(A^c) = 1 - P(A)$ where A^c is the complement event of A.

(iii) $P(A \setminus B) = P(A) - P(A \cap B)$.

Hence, deduce the *addition law* $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

(b) Find $P(A \cup B)$ and the *conditional probability* $P(A|B)$ when

(i) $A \subseteq B$.

(ii) A and B are *independent events.*

(iii) A and B are *mutually exclusive events.*

(c) A number is randomly selected between 1 and 600 inclusive. What is the probability that the number is a multiple of 12 but not a multiple of 20?
Find the conditional probability that the number is a multiple of 12 given that the number is a multiple of 20.

05. (a) Let a and b be two real numbers. Let x_1 and x_2 be two *distinct roots* of the equation $x^2 = ax + b$. If $f_n = cx_1^n + dx_2^n$ for some real numbers c and d, then prove that f_n satisfies the *difference equation* $f_n = af_{n-1} + bf_{n-2}$.

(b) Let f_n be the number of series which contain *ones and twos only*, where $n \in \mathbb{N}$.

If sum of f_n is equal to n.

(i) Write down first 4 terms of f_n.

(ii) Construct a *difference equation* satisfied by f_n for all $n \geq 3$.

Hence, show that $f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1} - 1$.

If the sum of the series which contain ones and twos only is 7, find the number of all possible series.

(iii) Find an *explicit formula* for f_n. Hence, show by considering $f_0 = 0$ as the empty series has the sum 0 that $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$.

06. (a) Let f be the function from $A = \{a, b, c, d\}$ to $B = \{1, 2, 3, 4, 5\}$ defined by $f(a) = 5$, $f(b) = 4$, $f(c) = 2$ and $f(d) = 3$.

(i) Show that f is an *injective function.*

Is f a *bijective function*? Justify your answer.

(ii) Find the number of functions that can be formed from A to B.

How many of those functions are *injective*?

(b) Let A, B and C be any three subsets of a universal set $\cup$.

Without using *Venn diagram*, prove the *De Morgan's laws.*

By stating the appropriate laws in set theory, show that $\overline{\left[A \cup (B \cap C)\right]} = \left(\overline{C} \cup \overline{B}\right) \cap \overline{A}$, where $\overline{A}$ is the complement of the set A.

(c) Show that $p^{\frac{n^2+n}{2}}$ number of $n \times n$ *symmetric matrices* exists over $\mathbb{Z}_p = \{0, 1, 2, \dots, p-1\}$.

How many of these matrices are *diagonal matrices*? Justify your answer.

07. Let n, m and r be three positive integers such that $1 \le r \le n$ and $1 \le r \le m$. Prove each of the following using ONLY the *combinatorial arguments*:

(a) ${}^{3n}C_2 - {}^{2n}C_2 - {}^{n}C_2 = 2n^2$, by counting the number of ways of selecting a teacher and a student from an assembly of n teachers and $2n$ students.

(b) $n \times {}^{n-1}C_{r-1} = r \times {}^{n}C_r$, by counting the number of ways of selecting a r - person committee and the committee chair from a board of n members.

(c) ${}^{n}C_r = {}^{n-2}C_r + 2\,{}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$, by counting the number of ways of selecting r members from a set of n members.

(d) ${}^{n}C_2 + {}^{m}C_2 + nm = {}^{n+m}C_2$, by counting the number of ways of selecting 2 persons from a group of n men and m women.

(e) ${}^{n}C_0 + {}^{n}C_1 + ... + {}^{n}C_n = 2^n$, by counting the number of ways of making a choice about the selection of each of the elements in a set of n elements.

08. (a) Briefly explain that the number of ways of selecting n *identical* objects from t *different* groups, where each group contains at least n objects of the similar kind, is the same as the number of *integer solutions* to $x_1 + x_2 + ... + x_t = n$ subject to $x_i \ge 0$ for $i \in \{1, 2, ..., t\}$.

(b) Find the number of *integer solutions* to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 10$, if each x_i is non-negative and at least one is zero.

(c) (i) Suppose the order in which the fruits are chosen does not matter and each fruit type is identical. How many ways are there to choose 5 fruits from a fruit shop containing Apple, Orange, Pears, Pineapple, Mango, Avocado, and Guava?

(ii) How many ways are there to place 10 *identical* balls into 8 different bowls?
If these 10 balls are *different*, find the number of ways to place them in those 8 bowls.

(iii) A person has 12 ten rupee coins and he wants to distribute it among 6 beggars so that each beggar will get at least one coin. Model this as an integer problem by clearly stating all necessary constrains. Hence, find the number of ways of distributing those coins to the beggars.