The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2017/2018
Pure Mathematics - Level 05
PEU5305 Complex Analysis I

Duration: Two Hours

Date: 21.09.2018 **Time: 09.30am - 11.30am**

Answer Only Four Questions.

1. (a) Check whether the set $S = \{z \in \mathbb{C} : |z - 1 - i| > 1\}$ is a region.
   (b) Let S be a non-empty subset of $\mathbb{C}$. Prove that, if a limit point of S does not belong to S, then it must be a boundary point of S.
   (c) Find the boundary points, exterior points and the closure of the set $S = \{z \in \mathbb{C} : |z - 1| \leq 1\} \setminus \{0\}$.

2. (a) i. Using the $\epsilon - \delta$ definition, show that $\lim_{z \to 1+i} z^2 = 2i$.
   ii. Let $f(z) = \frac{z^2}{|z|}$. Does $\lim_{z \to 0} f(z)$ exist? Justify your answer.
   (b) Find the limit of the sequence $\{z_n\} = \left\{ \frac{\sqrt{n} + i(n+1)}{n} \right\}$.
   (c) Use the ratio test to show that the series $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{n 7^n}$ is convergent.

3. (a) Let f is differentiable at z_0, where $z_0 \in \mathbb{C}$. Prove that f is continuous at z_0.
   (b) Let $f(z)$ be defined by $f(z) = \frac{x^3 + iy^3}{2x^2 + 3y^2}$ if $z = x + iy \neq 0$, where x and y are real, and $f(0) = 0$.
   i. At which points, the Cauchy-Riemann equations satisfied? Justify your answer.
   ii. At which points $f(z)$ is differentiable? Justify your answer.
   (c) Let $f(z)$ be analytic in a region G. Show that if $|f(z)|$ is constant in G, then $f(z)$ is constant in G.

4. (a) Let $f(z) = z^2 + \bar{z}$ and let C be the line segment from 0 to $1+i$. Evaluate $\int_C f(z)\, dz$.
   (b) Let C be a contour and let f be a function continuous on C. Also, let M be a non-negative number such that $|f(z)| \leq M$ for all z on C. Then prove that

$$\left| \int_C f(z)\, dz \right| \leq ML,$$

   where L is the length of the contour C.

(c) Let C be the semi-circle $z = 2e^{i\theta}$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Using part (b), show that

$$\left| \int_C \frac{Log z}{z^2} \right| \leq \frac{\pi}{2}\left(\ln 2 + \frac{\pi}{2}\right).$$

5. (a) State **Cauchy's theorem.**

   Using **Cauchy's theorem**, evaluate $\int_C \frac{Log(z+1)}{z-2} dz$, where C is the circle $|z| = \frac{1}{2}$, oriented counterclockwise.

   (b) State **Cauchy's integral formula.**

   Using **Cauchy's integral formula**, evaluate $\int_C \frac{\sin \pi z + \cos \pi z}{z^2 - 1} dz$, where C is the ellipse $|z-1| + |z+1| = 3$, oriented counterclockwise.

   (c) Apply **Cauchy's integral formula for derivative**, evaluate $\int_C \frac{e^z}{(z-3)(z-1+i)^2} dz$, where C is the circle $|z| = 2$, oriented counterclockwise.

6. (a) Find the **Laurent series** expansion of the function $f(z) = \frac{1}{z(z-1)(z-2)}$ in the annulus $1 < |z| < 2$.

   (b) State the **Residue theorem.**

   Using the **Residue theorem**, evaluate $\int_C \frac{1}{(z-1)^2(z+1)(z-2)} dz$, where C is the circle $|z-1| = 3$, oriented counterclockwise.

   (c) Use the **method of residues** to show that

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta} = \frac{2\pi}{a^2 - 1}; \qquad a > 1.$$

The End.