

The Open University of Sri Lanka
 B.Sc/B.Ed. Degree Programme
 Final Examination - 2017/2018
 Pure Mathematics - Level 05
 PUU3245 Complex Analysis I



Duration: Three Hours

Date: 21.09.2018

Time: 09.30am - 12.30pm

Answer Only Five Questions.

1. (a) Check whether the following sets are regions:
 - i. $\{z \in \mathbb{C} : |z - 1 - i| < 1\}$;
 - ii. $\{z \in \mathbb{C} : |z - 1 - i| > 1\}$.
- (b) By considering the sets $E_n = \left\{z \in \mathbb{C} : |z| < \frac{1}{n}\right\}$ for $n = 1, 2, 3, \dots$, show that an arbitrary intersection of open sets need not be open.
- (c) Find the boundary points, exterior points and the closure of the set $S = \{z \in \mathbb{C} : |z - 1| \leq 1\} \setminus \{0\}$.
2. (a) Using the $\epsilon - \delta$ definition, show that $\lim_{z \rightarrow 1+i} z^2 = 2i$.
- (b) Let $f(z) = \frac{z^2}{|z|}$. Does $\lim_{z \rightarrow 0} f(z)$ exist? Justify your answer.
- (c) Let $\{z_n\}$ be a sequence of complex numbers. Prove that if $\{z_n\}$ is Cauchy, then it is convergent.
- (d) Show that $\{i^n\}$ is not a Cauchy sequence.
3. (a) Let $\{z_n\}$ be a sequence of complex numbers and $z \in \mathbb{C}$. Show that if $\lim_{n \rightarrow \infty} z_n = z$, then $\lim_{n \rightarrow \infty} |z_n| = |z|$.
 Is the converse true? Justify your answer.
- (b) Find the limit of the sequence $\{z_n\} = \left\{\frac{\sqrt{n} + i(n+1)}{n}\right\}$.
- (c) Check whether each of the following series is convergent or divergent:
 - i. $\sum_{n=1}^{\infty} \frac{1 + in(-1)^n}{n^2}$;
 - ii. $\sum_{n=1}^{\infty} \frac{(n+i)(1+ni)}{n^2}$;
 - iii. $\sum_{n=0}^{\infty} \left(\frac{1}{2+i}\right)^n$.

4. (a) Let f be differentiable at z_0 , where $z_0 \in \mathbb{C}$. Prove that f is continuous at z_0 .
- (b) Let $f(z)$ be defined by $f(z) = \frac{x^3 + iy^3}{2x^2 + 3y^2}$ if $z = x + iy \neq 0$, where x and y are real, and $f(0) = 0$.
- At which points, the Cauchy-Riemann equations are satisfied? Justify your answer.
 - At which points $f(z)$ is differentiable? Justify your answer.
- (c) Let $f(z)$ be analytic in a region G . Show that if $|f(z)|$ is constant in G , then $f(z)$ is constant in G .

5. (a) Let $f(z) = z^2 + \bar{z}$ and let C be the line segment from 0 to $1+i$. Evaluate $\int_C f(z) dz$.
- (b) Let C be a contour and let f be a function continuous on C . Also, let M be a non-negative number such that $|f(z)| \leq M$ for all z on C . Then prove that

$$\left| \int_C f(z) dz \right| \leq ML,$$

where L is the length of the contour C .

- (c) Let C be the semi circle $z = 2e^{i\theta}$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Using part (b), show that

$$\left| \int_C \frac{\text{Log} z}{z^2} dz \right| \leq \frac{\pi}{2} \left(\ln 2 + \frac{\pi}{2} \right).$$

6. (a) State **Cauchy's theorem**.

Using **Cauchy's theorem**, evaluate $\int_C \frac{\text{Log}(z+1)}{z-2} dz$, where C is the circle $|z| = \frac{1}{2}$, oriented counterclockwise.

- (b) State **Cauchy's integral formula**.

Using **Cauchy's integral formula**, evaluate $\int_C \frac{\sin \pi z + \cos \pi z}{z^2 - 1} dz$, where C is the ellipse $|z-1| + |z+1| = 3$, oriented counterclockwise.

- (c) Apply **Cauchy's integral formula for derivative**, evaluate $\int_C \frac{e^z}{(z-3)(z-1+i)^2} dz$, where C is the circle $|z| = 2$, oriented counterclockwise.

7. (a) Find the **Taylor series expansion** of the function $f(z) = e^z$ for all $z \in \mathbb{C}$.

- (b) Find the **Laurent series expansion** of the function $f(z) = \frac{1}{z(z-1)(z-2)}$ in the annulus $1 < |z| < 2$.

- (c) State the **Residue theorem**.

Using the **Residue theorem**, evaluate $\int_C \frac{1}{(z-1)^2(z+1)(z-2)} dz$, where C is the circle $|z-1| = 3$, oriented counterclockwise.

8. For each of the following use the method of residues to show that

$$(a) \int_0^\pi \frac{d\theta}{(a + b \cos \theta)^2} = \frac{\pi a}{(a^2 - b^2)^{\frac{3}{2}}}; \quad 0 < b < a;$$

$$(b) \int_0^{2\pi} \frac{d\theta}{1 - 2a \cos \theta + a^2} = \frac{2\pi}{1 - a^2}; \quad |a| < 1.$$

The End.