The Open University of Sri Lanka
B.Sc./ B.Ed. Degree/ Continuing Education Programme
Level-04 Final Examination-2010/2011
PMU 2192/ PME 4192- Linear Algebra
Pure Mathematics

Duration: Two Hours.

Date: 10- 01- 2011.	**Time: 09.30 a.m. – 11.30 a.m.**

Answer FOUR questions only.

1. (i) Define a linear combination of vectors in a vector space.

   (ii) Define a subspace of a vector space.

   (iii) Show that the set of vectors $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x+y+2z=0 \right\}$ form a subspace of $\mathbb{R}^3$ and of dimension 2.

   (iv) For each of the following, determine whether W is a subspace of $\mathbb{R}^3$, if W consists vectors of the form

   (a) $\begin{pmatrix} a+b \\ c \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ b \\ c-1 \end{pmatrix}$ (c) $\begin{pmatrix} a \\ b \\ 3c+b \end{pmatrix}$ where $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$.

2. (i) Define a basis of a vector space.

   (ii) Determine whether each of the following set of vectors form a basis for the vector space $\mathbb{R}^3$ or not. Justify your answer.

   (a) (3, 0, -7) and (4, -2, 8).

   (b) (3, -1, 6), (2, 0, 9), (-5, 1, 0), and (7, 6, 5).

   (c) (1, 1, 1), (1, 2, 3) and 2, -1, 1).

   (d) (1, 1, 2), (1, 2, 5) and (5, 3, 4).

   (iii) Let S be the space spanned by the vectors

   $$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}.$$

   (a) Obtain a basis for S.

(b) Hence,determine the dimension of S.

3. (i) Which of the following are linear transformations?

(a) $L : V \to \mathbb{R}$ defined by

$$L(V) = \int_0^t V(t)\,dt.$$

(b) $F : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x\,y) = x\,y.$$

(c) $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x\,y) = (x^2, y^2).$$

(ii) Verify that the set of vectors $\left\{ v_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right),\ v_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right),\ v_3 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \right\}$ is an orthonormal basis for the Euclidean space $\mathbb{R}^3$.

(iii) Determine whether $(u, v) \in \mathbb{R}^2$ is an inner product or not when $\langle u, v \rangle = x_1 y_1 x_2 y_2$ where $u = (x_1, x_2),\ v = (y_1, y_2)$.

4. (i) Show that the following two matrices are equivalent.

$$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 4 & -1 & -2 \end{pmatrix}.$$

(ii) Find the inverse of the matrix B where

$$B = \begin{pmatrix} 2 & 5 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 3 & 6 & 3 & 2 \\ 4 & 12 & 0 & 8 \end{pmatrix}.$$

(iii) Find the non-singular matrices P and Q such that the normal form of A is PAQ where

$$A = \begin{pmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{pmatrix}.$$

(iv) Show that a system of equations $AX = B$ possesses a unique solution if the matrix A is non-singular.

5. (i) Define the rank of a matrix.

(ii) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 7 & 4 \\ 4 & 7 & 10 & 13 & 16 \\ 5 & 8 & 9 & 10 & 3 \end{pmatrix}.$$

(iii) For which rational numbers a and b does the following system have

(a) No solution?

(b) A unique solution?

(c) Infinitely many solutions?

Justify your answers.

$$\begin{aligned} x-2y+3z&=4 \\ 2x-3y+az&=5 \\ 3x-4y+5z&=b \end{aligned}.$$

(iv) If k is a non-zero scalar, prove that the characteristic roots of kA are k times the characteristic roots of A.

6. (i) State Cayley Hamilton theorem.

(ii) If matrix $A=\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ then find A^{-1}. Also, find the matrix B where

$B=A^8-5A^7+7A^6-3A^5-5A^3+8A^2-2A+I.$

(iii) Let $A=\begin{pmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{pmatrix}.$

Find an orthogonal matrix U such that $U^{-1}AU$ is diagonal.