The Open University of Sri Lanka
B.Sc. / B.Ed Degree Programme / Continuing Education Programme
Final Examination 2012/2013
Applied Mathematics – Level 04
APU 2143/APE4143 – Vector Calculus
Duration :- Two Hours.

Date :- 07.06.2013 Time:- 9.30 a.m. - 11.30 a.m.

Answer Four Questions Only.

1. (a) Define the limit of a two variable function $f(x, y)$ as $(x, y) \to (a, b)$; a and b being constants.

 (b) Define the continuity of a two variable function $f(x, y)$ at the point (a, b).

 (c) Discuss the continuity of the following functions at the point (0, 0).

 (i) $f(x, y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$

 (ii) $f(x, y) = \begin{cases} x\sin\dfrac{x+y}{4} & \text{for } (x, y) \neq (\pi, \pi) \\ \pi & \text{for } (x, y) = (\pi, \pi). \end{cases}$

2. (a) If $u = y^2 e^{\frac{y}{x}} + x^2 \tan^{-1}\left(\dfrac{x}{y}\right)$, prove that

 (i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u$ and (ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 2u$.

 (b) The base radius r of a right circular cone is increasing at the rate of 1.5mm/s while the perpendicular height h is decreasing at 6.0mm/s. Determine the rate at which the volume V is changing when $r = 12mm$ and $h = 24mm$.

 (c) A rectangular box has sides measured as 30mm, 40mm and 60mm. If these measurements are liable to be in error by $\pm 0.5mm$, $\pm 0.8mm$ and $\pm 1.0mm$ respectively, calculate the length of a diagonal of the box and the maximum possible error in the result.

3. (a) Prove that gradϕ is a vector normal to the contour surface $\phi(x,y,z) = c$, where c is a constant.

   (b) Determine the angle between the normals to the surface $xy = z^2$ at the points (4, 1, 2) and $(3, 3, -3)$.

   (c) If $\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ and $r = |\underline{r}|$ then prove that

   (i) $\underline{\nabla} r = \frac{\underline{r}}{r}$, (ii) $\underline{\nabla}(\ln r) = \frac{\underline{r}}{r^2}$, (iii) $\underline{\nabla}(r^n) = nr^{n-2}\underline{r}$ where n is an integer.

4. (a) State Gauss' Divergence theorem.

   (b) Verify the above theorem for the vector field $\underline{F} = x^2\underline{i} + z\underline{j} + y\underline{k}$ taken over the region bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 3,\ z = 0,\ z = 2$.

   (c) If S is any closed surface enclosing a volume V and $\underline{A} = ax\underline{i} + by\underline{j} + cz\underline{k}$, then evaluate $\iint_S \underline{A}.\underline{n}\, dS$.

5. (a) (i) State Stokes' theorem.

   (ii) Verify Stokes' theorem for the vector field $\underline{F} = (2x - y)\underline{i} + -yz^2\underline{j} - y^2z\underline{k}$, where S is the surface $x^2 + y^2 + z^2 = 1,\ z \geq 0$.

   (b) If $\underline{a}$ is a constant vector and $\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ then show that

   (i) curl $(\underline{a} \times \underline{r}) = 2\underline{a}$, (ii) curl $[(\underline{r}.\underline{r})\underline{a}] = 2(\underline{r} \times \underline{a})$.

6. (a) Suppose S is a plane surface lying in the xy–plane bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C (Pdx + Qdy)$ where P and Q are functions of x and y.

   (b) Verify the above result for the integral $\oint_C (3x + 4y)dx - xydy$, where C is the closed curve given by $x^2 + y^2 = 4$.

   (c) Show that the vector field $\underline{F} = xy^2ze^{-x^2z}\underline{i} - ye^{-x^2z}\underline{j} + \frac{x^2y^2}{2}e^{-x^2z}\underline{k}$ is irrotational. Find a scalar function ϕ such that $\underline{F} = \underline{\nabla}\phi$.