The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme /Continuing Education Programme
Final Examination 2013/2014
Applied Mathematics – Level 04
APU 2143/APE4143 – Vector Calculus
Duration :- Two Hours.

Date :- 14.06.2014 Time:- 1.30 p.m. - 3.30 p.m.

Answer Four Questions Only.

1. (a) Define the limit of a two variable function $f(x, y)$ as $(x, y) \to (a, b)$.

   (b) Define the continuity of a two variable function $f(x, y)$ at the point (a, b).

   (c) Discuss the continuity of each of the following functions at the point (0, 0).

   (i) $f(x, y) = \begin{cases} \dfrac{y^4}{x^4 + 3y^4} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$

   (ii) $f(x, y) = \begin{cases} \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy} & \text{for } (x, y) \neq (0, 0) \\ -5/2 & \text{for } (x, y) = (0, 0). \end{cases}$

2. (a) If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, t being a parameter, show that

   (i) $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\dfrac{\partial u}{\partial s}\right)^2 + \left(\dfrac{\partial u}{\partial t}\right)^2\right]$ and

   (ii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^{-2s}\left[\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right]$.

   (b) Find the maximum and minimum values of the function $f(x, y) = x^2 + y^2 + x^2y + 4$ and determine their nature.

3. (a) Prove that grad ϕ is a vector normal to the contour surface $\phi(x,y,z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by

$$\frac{x-x_0}{\left(\frac{\partial F}{\partial x}\right)_P} = \frac{y-y_0}{\left(\frac{\partial F}{\partial y}\right)_P} = \frac{z-z_0}{\left(\frac{\partial F}{\partial z}\right)_P}.$$

(ii) Show that the equation of the tangent plane to the elliptic paraboloid $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ at the point $P(x_0, y_0, z_0)$ can be written as $\frac{z+z_0}{c} = \frac{2xx_0}{a^2} + \frac{2yy_0}{b^2}$.

(c) Suppose that over certain region of space the electric potential $V(x, y, z)$ is given by $V(x, y, z) = 5x^2 - 3xy + xyz.$

(i) Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\underline{v} = \underline{i} + \underline{j} + \underline{k}$.

(i) In which direction does $V(x, y, z)$ change most rapidly.
(ii) What is the maximum rate of change at P

4. (a) State Gauss' Divergence theorem.

(b) Verify the above theorem considering for the vector field $\underline{F} = 3x\underline{i} + xy\underline{j} + 2xz\underline{k}$ and the closed region bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0$ and $z = 1$.

(c) Let $\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ and $r = |\,\underline{r}\,|$ then prove that

(i) $\underline{\nabla}\cdot\underline{r} = 3$, (ii) $\underline{\nabla}\cdot r\underline{r} = 4r$, (iii) $\underline{\nabla}^2 r^3 = 12r$

5. (a) (i) State Stokes' Theorem.

(ii) Verify Stokes' Theorem considering the vector field $\underline{F} = y\underline{i} + z\underline{j} + x\underline{k}$, and S, the hemisphere $x^2 + y^2 + z^2 = 1,\ \ y \geq 0$, oriented to the direction of the positive y-axis.

(b) Show that $\underline{F} = (1+xy)e^{xy}\underline{i} + \left(e^y + x^2 e^{xy}\right)\underline{j}$ is a conservative vector field. Then find a scalar function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

6. (a) Suppose that S is a plane surface lying in the xy –plane bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$.

(b) Verify the above result for the integral $\oint_C xy^2 dx - x^2 y dy$, where C is the circle given by $x^2 + y^2 = 4$ with counterclockwise direction.