The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme /Continuing Education Programme
Final Examination 2014/2015
Applied Mathematics – Level 04
APU 2143/APE4143 – Vector Calculus
Duration :- Two Hours.

Date :-09.05.2015 **Time:- 1.30 p.m. - 3.30 p.m.**

Answer Four Questions Only.

1. (a) State and sketch the domain of the function $f(x, y) = \sqrt{a^2 - x^2 - y^2}$.

   (b) Sketch the level curves of the function $f(x, y) = \sqrt{a^2 - x^2 - y^2}$.

   (c) Find the value of the following limits, if they exist:

   (i) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$, (ii) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$.

   (d) Discuss the continuity of the following function at the origin:

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

2. (a) If $z = f(x, y)$ where $x = r\cos\theta$ and $y = r\sin\theta$, prove that

   (i) $\frac{\partial r}{\partial x} = \cos\theta$ (ii) $\frac{\partial r}{\partial y} = \sin\theta$

   (iii) $\frac{\partial \theta}{\partial x} = \frac{\sin\theta}{r}$ (iv) $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

   Hence show that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{\partial z}{\partial \theta}\frac{\sin\theta}{r}$ and $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\theta + \frac{\partial z}{\partial \theta}\frac{\cos\theta}{r}$.

   (b) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D. Explain briefly how you could determine its nature.

   (c) Find the maximum and minimum values of the function $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$ and determine their nature.

3. (a) Definē grad ϕ as a vector (in a usual notation) and prove that grad ϕ is a vector normal to the contour surface $\phi(x,y,z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z)=0$ at the point $P(x_0, y_0, z_0)$ is given by $(x-x_0)\left(\frac{\partial F}{\partial x}\right)_P + (y-y_0)\left(\frac{\partial F}{\partial y}\right)_P + (z-z_0)\left(\frac{\partial F}{\partial z}\right)_P = 0$.

(ii) Find the equation of the tangent plane to the surface $x^2 - y^2 + z^2 = 4$ at the point $P(2, 1, -1)$.

(c) A fly is in a room in which the temperature T is given by $T(x, y, z) = x^2 + y^4 + 2z^2$. The fly is at the point $(1, 1, 1)$ and realizes that he is cold. In what direction should he fly to warm up as quickly as possible?

4. (a) State Gauss' Divergence theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = 2xy\underline{i} + yz^2\underline{j} + xz\underline{k}$ and S as the surfaces of the cuboid given by $0 \le x \le 2,\ 0 \le y \le 1$ and $0 \le z \le 3$.

(c) Show that $\underline{F} = (2x\cos y - 2z^3)\underline{i} + \left(3 + 2ye^z - x^2\sin y\right)\underline{j} + \left(y^2e^z - 6xz^2\right)\underline{k}$ is a conservative vector field. Then find a scalar function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

5. (a) State Stokes' Theorem.

(b) Verify Stokes' Theorem considering the vector field $\underline{F} = y^2\underline{i} + xy\underline{j} - xz\underline{k}$ and S as the hemisphere $x^2 + y^2 + z^2 = a^2$ and $z \ge 0$.

(c) If $\underline{a}$ is a constant vector, $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$ then show that

(i) $\underline{\nabla}\cdot\underline{r} = 3$, (ii) $grad(\underline{a}\cdot\underline{r}) = \underline{a}$, (iii) curl $(\underline{a} \times \underline{r}) = 2\underline{a}$, (iv) $\nabla^2 r^2 = 6$,

where the symbol $\underline{\nabla}$ has a usual meaning.

6. (a) Suppose that S is a plane surface lying in the xy-plane bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$.

(b) Verify the above result for the integral $\oint_C (xy + y^2)dx + x^2dy$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.