The Open University of Sri Lanka

B.Sc/B.Ed. Degree Programme

Final Examination – 2017/2018

Pure Mathematics - Level 04

PEU4301 – Real Analysis II

Duration: - Two hours

Date: 22.04.2019 **Time: 1:30 p.m. – 3:30 p.m.**

Answer **FOUR** questions only.

Q1)

a) Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ and $a \in \mathbb{R}$. Prove that f is continuous at a if and only if f is left-continuous at a and f is right-continuous at a.

b) Let $f(x) = \begin{cases} 5x+2, & \text{if } x \leq 0 \\ x^2, & \text{if } x > 0. \end{cases}$

Show that

(i) f is left-continuous at $x = 0$, and

(ii) f is not continuous at $x = 0$.

c) Let $f(x) = \frac{7x-2}{8x+9}$ for each $x > 0$. Prove that $\lim_{x \to +\infty} f(x) = \frac{7}{8}$.

Q2)

a) State the definition of uniform continuity of a function on an interval.

b) Let f be a function defined on $[a, b]$ such that f is uniformly continuous on $[a, b]$. Prove that f is continuous on $[a, b]$.

c) Let $f(x) = \frac{1}{x^2}$ for $x \neq 0$. Show that

(i) f is uniformly continuous on $[1, +\infty)$, and

(ii) f is not uniformly continuous on $(0, 1]$.

d) Let $f: (0, 1] \to \mathbb{R}$ be defined by $f(x) = \sin\frac{1}{x}$. Show that f is not uniformly continuous on $(0, 1]$.

Q3)

a) Let f, g and h be three real valued functions defined on an interval $(a, b) \subseteq \mathbb{R}$, except possibly at the point $c \in (a, b)$, such that $f(x) \leq g(x) \leq h(x)$ for each

$x \in (a,b) - \{c\}$, and $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$ for some $L \in \mathbb{R}$. Prove that $\lim_{x \to c} g(x) = L$.

b) Let F be a real valued function such that $F(x)$ is bounded on $[-a, a]$, where a is a positive real number. Prove that $\lim_{x \to 0} x^2 F(x) = 0$.

c) Find the following limits:

(i) $\lim_{x \to 0} x^2 e^{\sin\left(\frac{1}{x}\right)}$ and

(ii) $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q}^c \\ x^2, & \text{if } x \in \mathbb{Q} \end{cases}$

Q4)

a) Let f be a real value function defined on the open interval I and $a \in I$. State the $\varepsilon - \delta$ definition for differentiability at point a.

Let the function $f: (0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$, $x \in (0, +\infty)$. Show that f is differentiable at $x = 16$ and $f'(16) = {}^1/_8$.

b) Prove that if f is differentiable at point a, then f is continuous at a.

Show that the function f defined by $f(x) = \begin{cases} \frac{1}{x}\sin\left(\frac{1}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is not differentiable at point 0.

Q5) State the Rolle's theorem and the Mean-Value theorem for derivatives for a function f that is continuous on $[a, b]$ and differentiable on (a, b).

(i) Prove that if $f'(x) = 0$ for each $x \in (a, b)$ then $f(x)$ is a constant on (a, b).
(ii) Show that the equation $e^x + x^3 = 2$ has a unique real root.
(iii) Let $f(x) = x^3 - 4x$. Show that there is precisely one $a \in (-2, 1)$ which satifies the conclusion of the Mean-Value thorem on $[-2, 1]$.

Q6)

a) Let f be a real value function defined on an open interval (a, b) and let f has a local maximum at $c \in (a, b)$. Prove that if f is differentiable at point c, then $f'(c) = 0$.

b) Find the following limits if they exist:

(i) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ (ii) $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x}\right)$ (iii) $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$

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