The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme /Continuing Education Programme
Final Examination 2015/2016
Applied Mathematics – Level 04
APU 2143/APE4143 – Vector Calculus
Duration :- Two Hours.

Date :-08.07.2016 **Time:- 1.30 p.m. - 3.30 p.m.**

Answer Four Questions Only.

1. (a) State and sketch the domain of the function $f(x, y)=\dfrac{1}{\sqrt{16-x^2-y^2}}$.

   (b) Sketch the level curves of the function $f(x, y)=\dfrac{1}{\sqrt{16-x^2-y^2}}$.

   (c) Find the following limits, if they exist:

   (i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^2}$, (ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+y^2}$.

   (d) Discuss the continuity of the following function at the origin:

$$f(x, y)=\begin{cases} \dfrac{4xy^2}{x^2+y^2} & \text{if } (x, y)\neq(0, 0) \\ 0 & \text{if } (x, y)=(0, 0). \end{cases}$$

2. (a) If $z=f(x, y)$ where $x=\frac{1}{2}\left(u^2-v^2\right)$ and $y=uv$ then show that

   (i) $2\left(x\dfrac{\partial z}{\partial y}-y\dfrac{\partial z}{\partial x}\right)=u\dfrac{\partial z}{\partial v}-v\dfrac{\partial z}{\partial u}$,

   (ii) $\dfrac{\partial^2 z}{\partial u^2}+\dfrac{\partial^2 z}{\partial v^2}=\left(u^2+v^2\right)\left(\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}\right)$.

   (b) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D. Explain briefly how you could determine its nature.

   (c) Find the maximum and minimum values of the function $f(x, y)=x^4+xy^2-x^2y^2+3$ and determine their nature.

3. (a) Prove that grad ϕ is a vector normal to the contour surface $\phi(x,y,z) = c$, where c is a constant.

(b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point $P(x_0, y_0, z_0)$ is given by $(x-x_0)\left(\frac{\partial F}{\partial x}\right)_P + (y-y_0)\left(\frac{\partial F}{\partial y}\right)_P + (z-z_0)\left(\frac{\partial F}{\partial z}\right)_P = 0$.

(ii) Using the above result, find the equation of the tangent plane to the surface $F(x, y, z) = z\sqrt{x^2+y} + 2\frac{y}{z}$ at the point $P(1, 3, 2)$.

(c) The function $T(x, y, z) = x^2 + 2y^2 + 2z^2$ gives the temperature at each point in space. At the point P (1, 1, 1) in which direction should one go to get the most rapid increase in T.

4. (a) State Gauss' Divergence theorem.

(b) Verify the above theorem considering the vector field $\underline{F} = x\underline{i} + y\underline{j} + z\underline{k}$ and the surface S formed by the faces of the cuboid given by $0 \le x \le 1,\ 0 \le y \le 1$ and $0 \le z \le 1$.

(c) For what values of a and b is the vector field $\underline{F} = yz^2\underline{i} + (xz^2 + ayz)\underline{j} + (bxyz + y^2)\underline{k}$ be a conservative?. Using those values, find the corresponding scalar potential function ϕ such that $\underline{F} = \underline{\nabla}\phi$.

5. (a) State Stokes' Theorem.

(b) Verify Stokes' Theorem considering the vector field $\underline{F} = xy\underline{i} + z^2\underline{j} + xyz\underline{k}$ and S is the hemisphere $x^2 + y^2 + z^2 = a^2$ and $z \ge 0$.

(c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ be the position vector of the point (x, y, z) and $r = |\underline{r}|$. Then show that (for $r \ne 0$)

(i) $\underline{\nabla} r = \frac{\underline{r}}{r}$, (ii) $\underline{\nabla}(r^n) = nr^{n-2}\underline{r}$, where n is an integer (iii) $\underline{\nabla}(\ln r) = \frac{\underline{r}}{r^2}$ (iv) $\nabla^2 \ln r = \frac{1}{r^2}$.

6. (a) Suppose that S is a plane surface lying in the xy-plane bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$.

(b) Verify the above result for the integral $\oint_C (x^2+y^2)dx + (x+2y)dy$, where C is the closed curve of the region in the first quadrant bounded by $y = 0,\ x^2 + y^2 = 4$ and $x = 0$.