

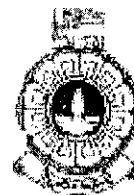
The Open University of Sri Lanka

B.Sc/B.Ed. Degree Programme – Level 04

Final Examination - 2017/2018

Pure Mathematics

PEU4316/PUU2143 – Differentiable Functions



Duration: - Two Hours.

Date: -22.04.2019

Time: - 1.30 p.m. – 3.30 p.m.

Answer 04 Questions.

(01). (a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$ for each $x \in \mathbb{R}$ and $c \in \mathbb{R}$. Using $\varepsilon - \delta$ definition prove that f is differentiable at c and that $f'(c) = 2c$.

(b) Let $g(x) = \begin{cases} x-1 & , x > 1 \\ 0 & , x \leq 1 \end{cases}$, for $x \in \mathbb{R}$. Prove that g is not differentiable at 1.

(c) Find a function f such that f is not differentiable at 1 and $|f|$ is differentiable at 1.

(02). (a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = 3x^2 + 5x + 7$. Using $\varepsilon - \delta$ definition. Show that f is differentiable at 0 that $f'(0) = 5$.

(b) Let f and g be functions and $c \in \mathbb{R}$ such that $(c - \delta_0, c + \delta_0) \subseteq \text{Domn}(f) \cap \text{Domn}(g)$ for some $\delta_0 > 0$ and both f, g are differentiable at c . Prove that $f + g$ is differentiable at c and that $(f + g)'(c) = f'(c) + g'(c)$.

(c) Find two functions f and g such that both are not differentiable at 3 but $f + g$ is differentiable at 3.

(03). (a) Let f be a function and $c \in \mathbb{R}$ such that $(c - \delta, c + \delta) \subseteq \text{Domn}(f)$ for some $\delta > 0$. Prove that if f is differentiable at c , then f is continuous at c . Is the converse true? Justify your answer.

(b) (i) Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by

$f(x) = \begin{cases} 0, & x \in \mathbb{R} \\ x, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Prove that the function is nowhere differentiable.

(ii) Find a function g define on \mathbb{R} such that g is differentiable at 0, but not differentiable at all other points.

(04). (a) Let $f(x) = ax^2 + bx + c$, $x \in \mathbb{R}$, a, b, c are real numbers such that $a \neq 0$

$$\text{Show that for each } x \in \mathbb{R}, f(x) - f\left(\frac{-b}{2a}\right) = a\left(x + \frac{b}{2a}\right)^2.$$

Deduce that

(i). f has a local minimum at $\frac{-b}{2a}$ if $a > 0$

(ii). f has a local maximum at $\frac{-b}{2a}$ if $a < 0$.

(b) Define $f: (-1, 1) \rightarrow \mathbb{R}$ by $f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right), & x \in (-1, 1) \setminus \{0\} \\ 0 & , x = 0 \end{cases}$

Show that f is differentiable at 0 and $f'(0) = 0$.

Is $f(0)$ a local maximum of f ? Justify your answer.

(c) Define $g: (-1, 1) \rightarrow \mathbb{R}$ by $g(x) = \begin{cases} -x + 1, & 0 \leq x < 1 \\ x + 1, & -1 < x < 0 \end{cases}$

Show that $g(0)$ is an absolute maximum of g

Is g differentiable at 0? Justify your answer.

(05). (a) State Rolle's Theorem.

State and deduce the Mean Value Theorem.

Find a function f define on $[-1, 1]$ such that f is continuous on $[-1, 1]$,

$f(-1) = f(1) = 0$, f is differentiable on $(-1, 1) \setminus \{0\}$ and there does not exist

$c \in (-1, 1)$ such that $f'(c) = 0$.

(b) Suppose f is a continuous function on (a, b) and f is differentiable on $(a, b) \setminus \{c\}$

where $c \in (a, b)$. Prove that if $\lim_{x \rightarrow c} f'(x) = l$, then f is differentiable at c and

$f'(c) = l$, where $l \in \mathbb{R}$. (Hint: use Generalized Mean Value Theorem)

(06). (a) State the Leibnitz' Theorem for higher derivatives.

Let $h(x) = e^x \cos x$, $x \in \mathbb{R}$. Find $h^{(5)}(x)$ for each $x \in \mathbb{R}$.

(b) Evaluate each of the following limit.

(i) $\lim_{x \rightarrow 1} \frac{\log(x)}{x-1}$

(ii) $\lim_{x \rightarrow 1^-} \frac{\sin(1-x)}{\sqrt{1-x}}$

(iii) $\lim_{x \rightarrow 0^+} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$