00122

The Open University of Sri Lanka

B.Sc/B.Ed. Degree Programme – Level 04

Final Examination - 2017/2018

Pure Mathematics

PEU4316/PUU2143 – Differentiable Functions

Duration: - Two Hours.

Date: -22.04.2019 **Time: - 1.30 p.m. – 3.30 p.m.**

Answer 04 Questions.

(01). (a) Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=x^2$ for each $x\in\mathbb{R}$ and $c\in\mathbb{R}$. Using $\varepsilon-\delta$ definition prove that f is differentiable at c and that $f'(c)=2c$.

(b) Let $g(x)=\begin{cases} x-1 & ,x>1 \\ 0 & ,x\le 1 \end{cases}$, for $x\in\mathbb{R}$. Prove that g is not differentiable at 1.

(c) Find a function f such that f is not differentiable at 1 and $|f|$ is differentiable at 1.

(02). (a) Let $f:\mathbb{R}\to\mathbb{R}$ be given by $f(x)=3x^2+5x+7$. Using $\varepsilon-\delta$ definition.Show that f is differentiable at 0 that $f'(0)=5$.

(b) Let f and g be functions and $c\in\mathbb{R}$ such that $(c-\delta_0,c+\delta_0)\subseteq \text{Domn}(f)\cap\text{Domn}(g)$ for some $\delta_0>0$ and both f,g are differentiable at c. Prove that $f+g$ is differentiable at c and that $(f+g)'(c)=f'(c)+g'(c)$.

(c) Find two functions f and g such that both are not differentiable at 3 but $f+g$ is differentiable at 3.

(03). (a) Let f be a function and $c\in\mathbb{R}$ such that $(c-\delta,c+\delta)\subseteq\text{Domn}(f)$ for some $\delta>0$. Prove that if f is differentiable at c, then f is continuous at c. Is the converse true? Justify your answer.

(b) (i) Show that the function $f:\mathbb{R}\to\mathbb{R}$ given by

$f(x)=\begin{cases} 0, & x\in\mathbb{R} \\ x, & x\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$. Prove that the function is nowhere differentiable.

(ii) Find a function g define on $\mathbb{R}$ such that g is differentiable at 0 ,but not differentiable at all other points.

(04). (a) Let $f(x)=ax^2+bx+c$, $x\in\mathbb{R}$, a,b,c are real numbers such that $a\neq 0$

Show that for each $x\in\mathbb{R}$, $f(x)-f\left(\frac{-b}{2a}\right)=a\left(x+\frac{b}{2a}\right)^2$.

Deduce that

(i). f has a local minimum at $\frac{-b}{2a}$ if $a>0$

(ii). f has a local maximum at $\frac{-b}{2a}$ if $a<0$.

(b) Define $f:(-1,1)\to\mathbb{R}$ by $f(x)=\begin{cases} x^4\sin\left(\frac{1}{x}\right), & x\in(-1,1)\setminus\{0\} \\ 0 & ,x=0\end{cases}$

Show that f is differentiable at 0 and $f'(0)=0$.

Is $f(0)$ a local maximum of f ?. Justify your answer.

(c) Define $g:(-1,1)\to\mathbb{R}$ by $g(x)=\begin{cases} -x+1 \ , & 0\leq x<1 \\ x+1 \ , & -1<x\leq 0\end{cases}$.

Show that $g(0)$ is an absolute maximum of g

Is g differentiable at 0 ? Justify your answer.

(05). (a) State Rolle's Theorem.
State and deduce the Mean Value Theorem.

Find a function f define on $[-1,1]$ such that f is continuous on $[-1,1]$, $f(-1)=f(1)=0$, f is differentiable on $(-1,1)\setminus\{0\}$ and there does not exist $c\in(-1,1)$ such that $f'(c)=0$.

(b) Suppose f is a continuous function on (a,b) and f is differentiable on $(a,b)\setminus\{c\}$ where $c\in(a,b)$. Prove that if $\lim_{x\to c} f'(x)=l$, then f is differentiable at c and $f'(c)=l$, where $l\in\mathbb{R}$. *(Hint: use Generalized Mean Value Theorem)*

(06). (a) State the Leibnitz' Theorem for higher derivatives.

Let $h(x) = e^x \cos x$, $x \in \mathbb{R}$. Find $h^{(5)}(x)$ for each $x \in \mathbb{R}$.

(b) Evaluate each of the following limit.

(i) $\lim_{x \to 1} \frac{\log(x)}{x-1}$ (ii) $\lim_{x \to 1^-} \frac{\sin(1-x)}{\sqrt{1-x}}$ (iii) $\lim_{x \to 0^+} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$