00135

The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination – 2017/2018
Pure Mathematics - Level 04
PUU2144 - Group Theory I

Duration: - Two hours

Date: 04.04.2019 **Time: 9:30 a.m. – 11:30 a.m.**

Answer **FOUR** questions only.

1) a) State a complete set of axioms satisfied by a group.
Let $G = \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. The operation "$\star$" on G is defined as

$$(a,b) \star (c,d) = (a+c, (-1)^c b + d), \forall\, (a,b), (c,d) \in G$$

Prove that $(G,\star)$ is a group.
Is G an Abelian group? Justify your answer.

b) Let G= $\{1,2,3,4,5,6\}$ be a group under the operation $\otimes_7$ defined by
$a \otimes_7 b = r, 0 < r < 7$; where r is the remainder when ordinary multiplication ab is divided by 7.

i) Show that $(G, \otimes_7)$ is cyclic.
ii) How many generators are there of the cyclic group $(G, \otimes_7)$? Justify your answer.

2) a) Prove that the order of a finite group is equal to the order of its generator.

b) Prove or disprove the following statements.
i) If H_1 and H_2 are two subgroups of a group G, then $H_1 \cap H_2$ is a subgroup of G.
ii) If H_1 and H_2 are two subgroups of a group G, then it is necessary true that $H_1 \cup H_2$ is a subgroup of G.
iii) A subgroup of a cyclic group is cyclic.

3) a) Let $G = \{(a,b): a,b \in \mathbb{C}, b \neq 0\}$
i) Show that G is a non abelian group under the operation $\star$ defined by

$$(a,b) \star (c,d) = (a+bc, bd), \forall\, (a,b), (c,d) \in G$$

ii) Let $H = \{(na-n, a): a \neq 0, n \in \mathbb{Z}$ is a fixed positive integer$\}$
Show that H forms a subgroup of G.

b) Find the order of each element in the group $(\mathbb{Z}_6, \oplus_6)$.
Does $\mathbb{Z}_6$ form a cyclic group? Justify your answer.

4) a) Prove that any two right cosets of a sub group of a group are either disjoint or identical

b) Let G be the group of all permutations of degree 3 on three symbols 1, 2 and 3. Then the elements of G are the permutations:
$f_1 = (1),\ f_2 = (1\ 2),\ f_3 = (2\ 3),\ f_4 = (3\ 1),\ f_5 = (1\ 2\ 3),\ f_6 = (1\ 3\ 2)$.
Form the right cosets of H in G by considering $H = \{f_1, f_2\}$.

c) Explain, with examples, the terms "even permutation" and "odd permutation".
Determine which of the following permutations are even and which are odd.

i) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix}$

iii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$

iv) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 3 & 6 & 1 & 7 & 9 & 8 \end{pmatrix}$

5) Prove or disprove each of the following statements

a) If $f : G \to G'$ is a homomorphism, then $f(x^{-1}) = [f(x)]^{-1}, \forall\, x \in G$, where e and e' are identity elements of G and G' respectively.

b) Let $G = (\mathbb{Z} \times \mathbb{Z}, +)$ and $G' = (\mathbb{Z}, +)$ be groups. The map $f : G \to G'$ defined by $f(m, n) = m$ is a homomorphism from G to G'.

c) The map $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2, \forall\, x \in \mathbb{R}$. Then $(\mathbb{R}, \cdot) \cong (\mathbb{R}, \cdot)$.

d) If a is a fixed element of a group G, then the map $f : G \to G$ such that $f(x) = axa^{-1}, \forall\, x \in G$ is an isomorphism of G to itself.

e) The additive group G of integers is isomorphic to the multiplicative group G' where $G' = \{\dots, 3^{-3}, 3^{-2}, 3^{-1}, 1, 3^1, 3^2, 3^3, \dots\}$

6) Let G, G' be two groups and $f : G \to G'$ be a homomorphism. Define the kernel of f (Ker f). Prove that:

a) Kernel of f is a normal subgroup of G.

b) f is one to one if and only if $Ker\ f = \{e\}$, where e is the identity of G.

c) f is an isomorphism if and only if $f(G) = G'$ and $Ker\ f = \{e\}$.