The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme – Level 05
Final Examination – 2006/2007
Pure Mathematics
PMU 3292/PME 5292 – Group Theory & Transformation – Paper I

Duration :- Two and Half Hours

Date :- 03-11-2006 **Time :- 9.30 a.m. – 12.00 noon.**

Answer Four Questions Only.

01. (a) Prove that the set G of all matrices $A\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $\alpha \in R$, together with matrix multiplication is an abelian group.

(b) Show that the members of each of the following operation table form a group:

*	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

02.(a) If $(G, *)$ is a group then prove that $x = e$ is the unique solution of the group equation $x * x = x$.

(b) Given that $a^2 = e$ for every element a of the group $(G, *)$. Show that the group G must be commutative.

(c) Prove that a group $(G, *)$ is commutative if and only if $(a * b)^{-1} = a^{-1} * b^{-1}$.

03. Let M be the set of all 2×2 non-complex matrices with matrix multiplication as binary operation. Determine which of the following subsets are groups: Justify your answers.

(a) The subset of all real matrices with only positive real numbers as entries.

(b) The subset of all matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $ac \neq 0$.

(c) The subset of all matrices of form $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ with $bc \neq 0$.

(d) The subset of all matrices of the form $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$.

04.(a) Show that $H = \{3n \mid n \in \mathbb{Z}, n \geq 0\}$ is not a subgroup of the group $(\mathbb{Z}, +)$.

(b) Let $(R, +)$ be the additive group of all real numbers and the set $H = \{x \mid 0 < x < 1\}$. Show that H is not a subgroup of $(R, +)$.

(c) If H is a subgroup of a group G, then prove that aHa^{-1} is a subgroup of G for some $a \in G$.

05.(a) G is the group of 2×2 complex non-singular matrices with matrix multiplication as the binary operation defined on G. Find the order of each of the following elements.

(i) $\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ (ii) $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

(b) Find the order of 3 of the group of non-zero integers modulo 5 under the operation $\odot_5$.

(c) If $(G, *)$ is a group and $a \in G$, then prove that $O(a^{-1}) = O(a)$, where $O(a)$ is the order of element a.

06.(a) If a is the generator of the cyclic group $\langle G, \cdot \rangle$, then prove that a^{-1} is also its generator.

(b) Which of the following groups are cyclic? List the generators of the cyclic groups.

(i) $(Z, +)$ (ii) $(Q, +)$

(iii) $(Q^{+}, \cdot)$ (iv) $(6Z, +)$

(v) $\{6^{n} \mid n \in Z\}$.