THE OPEN UNIVERSITY OF SRI LANKA

BACHELOR OF TECHNOLOGY HONORS IN ENGINEERING – LEVEL 04

FINAL EXAMINATION – 2015/2016

MPZ4230 – ENGINEERING MATHEMATICS II

DURATION: THREE (03) HOURS

Date: 30th November 2016 **Time: 0930hrs – 1230hrs**

Instructions:

- Answer only six (6) questions.
- State any assumptions you required.
- This paper contains five (5) pages.
- Show all your workings.
- All symbols are in standard notation.

1. a) Let $u = r^5$ such that $r^2 = x^2 + y^2 + z^2$. Prove the followings,

   i. $\frac{\partial^2 u}{\partial x^2} = 5r^3 \left\{ r \frac{\partial^2 r}{\partial x^2} + 4 \left(\frac{\partial r}{\partial x}\right)^2 \right\}$

   ii. $r \frac{\partial r}{\partial x} - x = 0,$

   iii. $r \frac{\partial^2 r}{\partial x^2} = 1 - \left(\frac{\partial r}{\partial x}\right)^2,$

   Hence deduce that

   $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 30r^3.$ [35%]

   b) Find and classify the critical points of the following function.

   $f(x,y) = 2x^3 - 6x^2y + 18y^3 + 3x^2.$ [25%]

   c) Let $\underline{F} = (y^2 + z)\underline{i} + (2xy + z^2)\underline{j} + (2yz + x)\underline{k}.$

   i. Show that the vector field $\underline{F}$ is conservative.

   ii. Find the scalar potential ϕ such that $\underline{F} = grad\,\phi$. [40%]

2. a) i. State the Cauchy Riemann equation for the analytic function $f(z) = u + iv$. Hence show that u is a harmonic function. [10%]

ii. Verify that $u = x^3 - 3xy^2 - 5y$ is a harmonic function. [10%]

iii. Find the analytic function $f(z) = u + iv$, for $u = x^3 - 3xy^2 - 5y$. [10%]

b) i. Sketch the region of the right half of the circle $|z| = 1$ from $z = -i$ to $z = i$. [05%]

ii. Evaluate $\int_c \frac{1+z}{z}\, dz$, where c is stated as above in (i). [20%]

c) i. State Cauchy's integral Formula. Hence find $\oint_c \frac{e^z}{z-i\pi} dz$, $c: |z| = 4$ [15%]

ii. Evaluate the following integral using the residue theorem. [30%]

$$\int_c \frac{3z+1}{z^2-9}\, dz, \quad c: |z| = 4$$

3. A company has the following data on its profits and advertising expenditure over the last ten years:

Year	1	2	3	4	5	6	7	8	9	10
Advertising expenditure (Rs. In millions)	0.52	0.61	0.63	0.70	0.70	0.75	0.79	0.82	0.86	0.9
Profits (Rs. In millions)	11.4	12.1	12.6	13.4	13.9	14.5	14.8	15.2	15.9	16.4

a) If profit change according to the advertising expenditure, identify the independent variable and the dependent variable. [05%]

b) Determine the value of the Pearson correlation coefficient, correct to three decimal places. Interpret the Pearson correlation coefficient. [10%]

c) Find the best fit line for the data given.
Interpret the slope coefficient of the regression line. [30%]

d) Forecast the profits for the next year if an advertising budget of Rs. 950,000 is allocated. [10%]

e) Following table represents some summary statistics for simple linear regression model from the above data. Fill in the blanks of the following table. [15%]

Sources of variation	**Sum of squares**	**df**	**Mean square**	**F value**
Regression	24.071	1	.....	.....
Residul (error)	.....	.....	.....	
Total	24.396	9		

f) Test the significance of the regression model at 5% level of significance using above table. [30%]

4. a) Following are the weights recorded (in grams) of the contents of 12 boxes of cereal that are randomly selected from a filling process to check the weight.

No.	1	2	3	4	5	6	7	8	9	10	11	12
Weight (in grams)	506	508	499	503	504	510	497	512	514	505	493	496

Assume that the weight of the each box is a normally distributed random variable.

i. Determine 95% confidence interval for the mean weight of this process. Interpret this confidence interval. [25%]

ii. State the confidence limits. [05%]

b) A group of 10 students coming from different backgrounds were enrolled at a certain institute and their knowledge was tested through a test paper. After one year of special teaching method under the same environment, it is expected that they have improved their knowledge. To confirm this another test paper was given for these students and the marks obtained were recorded.

Student	Marks before special teaching method	Marks after special teaching method
1	80	89
2	12	43
3	18	51
4	90	90
5	30	68
6	50	72
7	70	81
8	10	40
9	20	60
10	95	97

i. Test using an appropriate method whether this special teaching method has helped in improving student's knowledge. (Use 5% level of significance) [65%]

ii. State the assumptions that you made. [05%]

5. Let $\frac{dy}{dx} = \frac{2}{x}\, y + x^2 e^x$, with exact solution $y(x) = x^2(e^x - e)$ where $0 \leq x \leq 2$ and $y(1) = 0$.

   a) Find $y(1.02)$ using,

      i. Euler's method, [20%]

      ii. Taylor series method with order three, [35%]

      iii. Runge – Kutta Fourth order method. [35%]

      The Fourth order Runge - Kutta formula for solving the differential equation

      $\frac{dy}{dx} = f(x, y)$ are

      $$k_1 = hf(x_m, y_m)$$

      $$k_2 = hf\left(x_m + \frac{1}{2}h, y_m + \frac{1}{2}k_1\right)$$

      $$k_3 = hf\left(x_m + \tfrac{1}{2}h, y_m + \tfrac{1}{2}k_2\right)$$

      $$k_4 = hf(x_m + h, y_m + k_3)$$

      $$y_{m+1} = y_m + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

   b) Find the exact value of $y(1.02)$. [10%]

6. a) State the classification for a partial differential equation in elliptic, parabolic and hyperbolic form. [10%]

   b) Using finite difference method, (Assuming step size on x axis $h = 0.1$ and y axis $k = 0.1$ respectively) solve the wave equation $u_{tt}(x, t) = u_{xx}(x, t)$ for $0 \leq x \leq 1$ and $0 \leq t \leq 0.5$ with the boundary conditions $u(0, t) = 0$ and $u(1, t) = 0$ for $0 \leq t \leq 0.5$, and the initial conditions,

   $u(x, 0) = \sin 2\pi x \quad 0 \leq x \leq 1$

   $u_t(x, 0) = \sin \pi x \quad 0 \leq x \leq 1$

   Hence compute the approximation $u(x, t)$ for the given region. Answer should be up to at least five (5) decimal places. [90%]

7. An electromagnetic field of 500 volts is in series with a 20 ohms resistor, an inductor of $1/0.25\ H$ and a 0.008 F capacitor. At $t = 0$, charge and current is zero.

   a) Find charge and current at any time t. [75%]

   b) Indicate the transient and steady state terms in charge and current [15%]

   c) Find the charge and current after long time. [10%]

   You may assume that,

$$\left[\begin{array}{c} \text{Kirchhoff's voltage law for series circuit yield} \\ V_L + V_R + V_C = V_0, \quad \text{where} \\ V_L = L\frac{di}{dt}, \quad V_R = iR, \quad V_C = \frac{1}{C}\int idt, \quad i = \frac{dq}{dt} \end{array}\right]$$

8. a) Show that the set of vectors (u_1, u_2, u_3) of $\mathbb{R}^3$ is linearly independent over $\mathbb{R}$, where $u_1 = (1,2,2), u_2 = (1,-2,2)$ and $u_3 = (1,0,1)$. [25%]

   b) Construct an orthogonal basis for $\mathbb{R}^3$ out of above (u_1, u_2, u_3) by the gram-Schmidt process. [25%]

   c) For the following linear transformation $T: V \to W$, Find a basis for $\ker(T)$ and basis for range (T). Verify the relation $\dim\{\ker(T)\} + \dim\{\text{range}(T)\} = \dim V$ [50%]

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ -2 & 3 & 13 \\ 3 & -3 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

9. Let $A = \begin{bmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{bmatrix}$

   a) Find the eigenvalues of A and corresponding unit eigenvectors. [30%]

   b) Find an orthogonal matrix P such that $D = P^{-1}AP$, where D is a diagonal matrix. [30%]

   c) Using part (b) prove that $D = P^{-1}A^{-1}P$ and $D^n = P^{-1}A^nP$, where n is a positive integer. [10%]

   d) Derive the quadratic functions of $Q(x) = x^TAx$ and $Q(y) = y^TDy$,

   where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, A and D as above. [15%]

   e) Show that $P^{-1} = P^T$, and hence show that $Q(x) = Q(y)$, where $y = P^Tx$. [15%]

-END-

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Formula sheet:-

$$\frac{\sum_{i=1}^{n}(x_i-\overline{x})(y_i-\overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\overline{x})^2\sum_{i=1}^{n}(y_i-\overline{y})^2}}$$

$$\frac{\sum x_i y_i - n\overline{x}\overline{y}}{\sum x_i^2 - n\overline{x}^2}$$

$$\frac{1}{n-2}\left[S_{yy}-\frac{S_{xy}^2}{S_{xx}}\right]$$

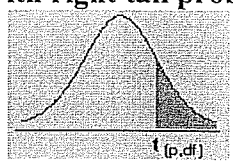
$$\left(\overline{X}-Z_{\frac{\alpha}{2}}\times\frac{\sigma}{\sqrt{n}}\,,\ \overline{X}+Z_{\frac{\alpha}{2}}\times\frac{\sigma}{\sqrt{n}}\right)$$

$$\left(\overline{X}-t_{n-1}\left(\frac{\alpha}{2}\right)\times\frac{S}{\sqrt{n}}\,,\ \overline{X}+t_{n-1}\left(\frac{\alpha}{2}\right)\times\frac{S}{\sqrt{n}}\right)$$

$$\left(\overline{X}-Z_{\frac{\alpha}{2}}\times\frac{\sigma}{\sqrt{n}}\times\sqrt{\frac{N-n}{N-1}}\ ,\ \overline{X}+Z_{\frac{\alpha}{2}}\times\frac{\sigma}{\sqrt{n}}\times\sqrt{\frac{N-n}{N-1}}\right)$$

$$\left(\overline{X}-t_{n-1}\left(\frac{\alpha}{2}\right)\times\frac{S}{\sqrt{n}}\times\sqrt{\frac{N-n}{N-1}},\ \overline{X}+t_{n-1}\left(\frac{\alpha}{2}\right)\times\frac{S}{\sqrt{n}}\times\sqrt{\frac{N-n}{N-1}}\right)$$

t table with right tail probabilities



df\p	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0005
1	0.324920	1.000000	3.077684	6.313752	12.70620	31.82052	63.65674	636.6192
2	0.288675	0.816497	1.885618	2.919986	4.30265	6.96456	9.92484	31.5991
3	0.276671	0.764892	1.637744	2.353363	3.18245	4.54070	5.84091	12.9240
4	0.270722	0.740697	1.533206	2.131847	2.77645	3.74695	4.60409	8.6103
5	0.267181	0.726687	1.475884	2.015048	2.57058	3.36493	4.03214	6.8688
6	0.264835	0.717558	1.439756	1.943180	2.44691	3.14267	3.70743	5.9588
7	0.263167	0.711142	1.414924	1.894579	2.36462	2.99795	3.49948	5.4079
8	0.261921	0.706387	1.396815	1.859548	2.30600	2.89646	3.35539	5.0413
9	0.260955	0.702722	1.383029	1.833113	2.26216	2.82144	3.24984	4.7809
10	0.260185	0.699812	1.372184	1.812461	2.22814	2.76377	3.16927	4.5869
11	0.259556	0.697445	1.363430	1.795885	2.20099	2.71808	3.10581	4.4370
12	0.259033	0.695483	1.356217	1.782288	2.17881	2.68100	3.05454	4.3178
13	0.258591	0.693829	1.350171	1.770933	2.16037	2.65031	3.01228	4.2208
14	0.258213	0.692417	1.345030	1.761310	2.14479	2.62449	2.97684	4.1405
15	0.257885	0.691197	1.340606	1.753050	2.13145	2.60248	2.94671	4.0728
16	0.257599	0.690132	1.336757	1.745884	2.11991	2.58349	2.92078	4.0150
17	0.257347	0.689195	1.333379	1.739607	2.10982	2.56693	2.89823	3.9651
18	0.257123	0.688364	1.330391	1.734064	2.10092	2.55238	2.87844	3.9216
19	0.256923	0.687621	1.327728	1.729133	2.09302	2.53948	2.86093	3.8834
20	0.256743	0.686954	1.325341	1.724718	2.08596	2.52798	2.84534	3.8495
21	0.256580	0.686352	1.323188	1.720743	2.07961	2.51765	2.83136	3.8193
22	0.256432	0.685805	1.321237	1.717144	2.07387	2.50832	2.81876	3.7921
23	0.256297	0.685306	1.319460	1.713872	2.06866	2.49987	2.80734	3.7676
24	0.256173	0.684850	1.317836	1.710882	2.06390	2.49216	2.79694	3.7454
25	0.256060	0.684430	1.316345	1.708141	2.05954	2.48511	2.78744	3.7251
26	0.255955	0.684043	1.314972	1.705618	2.05553	2.47863	2.77871	3.7066
27	0.255858	0.683685	1.313703	1.703288	2.05183	2.47266	2.77068	3.6896
28	0.255768	0.683353	1.312527	1.701131	2.04841	2.46714	2.76326	3.6739
29	0.255684	0.683044	1.311434	1.699127	2.04523	2.46202	2.75639	3.6594
30	0.255605	0.682756	1.310415	1.697261	2.04227	2.45726	2.75000	3.6460
inf	0.253347	0.674490	1.281552	1.644854	1.95996	2.32635	2.57583	3.2905

Probabilities for the standard normal distribution

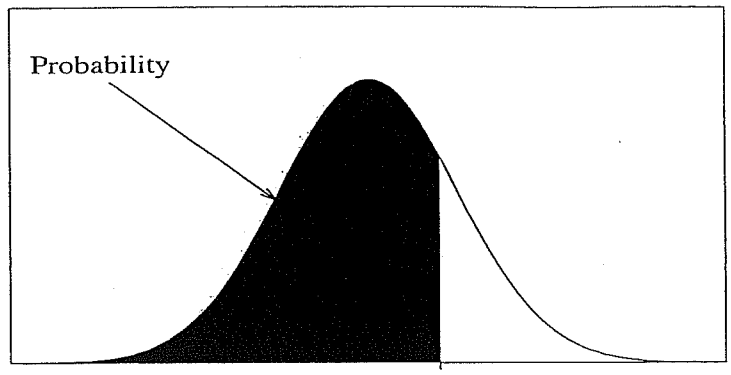
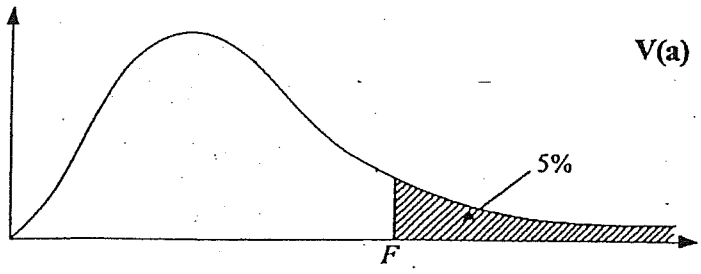


Table entry for z is the probability lying to the left of z

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

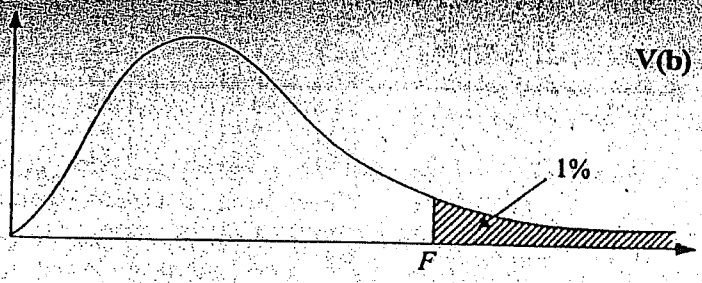
V(a)



5 ප්‍රතිශතයට අනුරූප F ව්‍යාප්තිය
F பரம்பலின் 5 சதவீத புள்ளிகள்
5 percent points of the F distribution

n_2 \ n_1 =	1	2	3	4	5	6	7	8	10	12	24
2	18.5	19.0	19.20	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.79	8.74	8.64
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	5.96	5.91	5.77
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.74	4.68	4.53
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.06	4.00	3.84
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.64	3.57	3.41
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.35	3.28	3.12
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.14	3.07	2.90
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	2.98	2.91	2.74
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.75	2.69	2.51
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.54	2.48	2.29
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.35	2.28	2.08
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.25	2.18	1.98
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.16	2.09	1.89
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.08	2.00	1.79
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	1.99	1.92	1.70

V(b)



1 ප්‍රතිශතයට අනුරූප F ව්‍යාප්තිය
F பரம்பலின் 1 சதவீத புள்ளிகள்
1 percent points of the F distribution

n_2 \ n_1 =	1	2	3	4	5	6	7	8	10	12	24
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.2	27.1	26.6
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.5	14.4	13.9
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.1	9.89	9.47
6	13.7	10.98	9.78	9.15	8.75	8.47	8.26	8.10	7.87	7.72	7.31
7	12.3	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.62	6.47	6.07
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.81	5.67	5.28
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.26	5.11	4.73
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.85	4.71	4.33
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.30	4.16	3.78
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.80	3.67	3.29
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.37	3.23	2.86
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.17	3.03	2.66
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	2.98	2.84	2.47
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.80	2.66	2.29
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.63	2.50	2.12