



Duration :- Two and Half Hours.

Date :- 26.06.2007

Time:- 10.00 a.m. – 12.30 p.m.

Answer FOUR Questions only.

01.(i) Consider the step function f defined on $[a, b]$ with $a = s_0 < s_1 < s_2 = b$ given by $f(s_i) = c_i$ for each $i = 0, 1, 2$, $f(x) = \alpha_1$ for each $x \in (s_0, s_1)$ and $f(x) = \alpha_2$ for each $x \in (s_1, s_2)$. Let $\varepsilon > 0$ be arbitrary and $M = \max\{|c_0|, |c_1|, |c_2|, |\alpha_1|, |\alpha_2|\}$. Suppose that

$$0 < \delta < \min\left\{\frac{s_1 - a}{2}, \frac{b - s_1}{2}, \frac{\varepsilon}{8(M+1)}\right\}.$$

Show that

(a) $\sigma = (a, a + \delta, s_1 - \delta, s_1 + \delta, b - \delta, b)$ is a partition of $[a, b]$.

(b) $U(f, \sigma) - L(f, \sigma) < \varepsilon$.

Deduce that f is Riemann integrable on $[a, b]$.

(ii) Now consider the step function g defined on $[a, b]$ with $a = s_0 < s_1 < s_2 < \dots < s_n < s_{n+1} = b$ given by $g(s_i) = c_i$, for each $i = 0, 1, 2, \dots, n + 1$, $g(x) = \alpha_i$ for each $x \in (s_{i-1}, s_i)$ and for each $i = 1, 2, \dots, n + 1$. Use the Riemann criterion to prove that g is Riemann integrable on $[a, b]$.

02. Let f, g and h be bounded functions defined on $[0, 1]$.

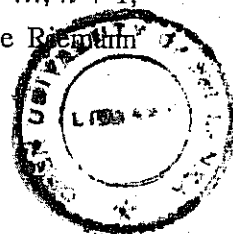
(i) Find a function f such that f^2 is Riemann integrable on $[0, 1]$, but f is not Riemann integrable $[0, 1]$. Justify your answer.

(ii) Suppose that the addition $g + h$ of g and h , is Riemann integrable on $[0, 1]$. Does it follow that product gh of g and h is Riemann integrable on $[0, 1]$? Justify your answer.

(iii) Suppose that there exists $\sigma \in P[0, 1]$ for each $\varepsilon > 0$ such that $U(f, \sigma) - L(f, \sigma) < \varepsilon$.

Is f Riemann integrable on $[0, 1]$? Justify your answer.

Give an example of a function f such as above such that $f(1/2) = 2007$.



03.(i) By dissecting interval $[1, 2]$ into n subintervals in geometric progression such that $1 < r < r^2 < \dots < r^n = 2$ and using the definition of Riemann integral, prove that

$$\int_1^2 x^4 dx = \frac{31}{5}. \text{ Hint: } \lim_{x \rightarrow 1} \frac{x-1}{x^n - 1} = \frac{1}{n}.$$

(ii) Deduce that $\lim_{n \rightarrow \infty} \frac{1}{n^5} \sum_{k=1}^n (n+k)^4 = \frac{31}{5}$.

(iii) Construct a function f on $[1, 2]$ such that the upper integral of f on $[1, 2]$, $\int_1^2 f(x) dx = \frac{31}{5}$

$$\text{and the lower integral of } f \text{ on } [1, 2], \int_1^2 f(x) dx = \frac{5}{31}.$$

04. (i) Let F be a differentiable function with a Riemann integrable derivative F' on $[a, b]$. Prove that

$$\int_a^x F'(t) dt = F(x) - F(a) \text{ for each } x \in [a, b].$$

(ii) Suppose that f is Riemann integrable on $[0, 1]$. Let $F(x) = \int_0^x f(t) dt$ for each $x \in [0, 1]$.

Is F differentiable on $[0, 1]$? Justify your answer.

05. Let n be a positive integer greater than 1.

(i) Prove that if f is continuous and increasing on $[1, \infty)$, then

$$\int_1^n f(x) dx \leq \sum_{k=2}^n f(k) \leq \int_2^{n+1} f(x) dx.$$

(ii) Deduce that

$$n \ln n - n + 1 \leq \ln n! \leq (n+1) \ln(n+1) - n + 1 - 2 \ln 2.$$

(iii) Prove that

$$\frac{n^n}{n!} \leq \exp n \leq \frac{(n+1)^{n+1}}{n!}.$$

06. Let f be a positive bounded function defined on $[2, \infty)$ such that f is Riemann integrable on $[2, k]$ for each $k > 2$.

(i) Suppose $\lim_{x \rightarrow \infty} x^2 f(x) = 0$. Prove that $\int_2^{\infty} f(x) dx$ converges.

(ii) Does $\lim_{x \rightarrow \infty} x f(x) = 0$ imply that f is Riemann integrable on $[2, \infty)$? Justify your answer.

(iii) Find a function f such that $\lim_{x \rightarrow \infty} f(x) = 0$ and f is not Riemann integrable on $[2, \infty)$. Justify your answer.