The Open University of Sri Lanka
B.Sc. Degree Programme – Level 04
Final Examination 2006/2007
Pure Mathematics
PMU 2195/PME 4195 – Theory of Integration

Duration :- Two and Half Hours.

Date :- 26.06.2007 **Time:- 10.00 a.m. – 12.30 p.m.**

Answer FOUR Questions only.

01.(i) Consider the step function f defined on $[a, b]$ with $a = s_0 < s_1 < s_2 = b$ given by $f(s_i) = c_i$ for each i = 0, 1, 2, $f(x) = \alpha_1$ for each $x \in (s_0, s_1)$ and $f(x) = \alpha_2$ for each $x \in (s_1, s_2)$. Let $\varepsilon > 0$ be arbitrary and $M = \max\{|c_0|, |c_1|, |c_2|, |\alpha_1|, |\alpha_2|\}$. Suppose that $0 < \delta < \min\left\{\frac{s_1 - a}{2}, \frac{b - s_1}{2}, \frac{\varepsilon}{8(M+1)}\right\}$.

Show that

(a) $\sigma = (a, a+\delta, s_1 - \delta, s_1 + \delta, b - \delta, b)$ is a partition of $[a, b]$.

(b) $U(f, \sigma) - L(f, \sigma) < \varepsilon$.

Deduce that f is Riemann integrable on $[a, b]$.

(ii) Now consider the step function g defined on $[a, b]$ with $a = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = b$ given by $g(s_i) = c_i$, for each i = 0, 1, 2, ..., n + 1, $g(x) = \alpha_i$ for each $x \in (s_{i-1}, s_i)$ and for each i = 1, 2, ..., n + 1. Use the Riemann criterion to prove that g is Riemann integrable on $[a, b]$.

02. Let f, g and h be bounded functions defined on [0, 1].

(i) Find a function f such that f^2 is Riemann integrable on [0, 1], but f is not Riemann integrable [0, 1]. Justify your answer.

(ii) Suppose that the addition $g + h$ of g and h, is Riemann integrable on [0, 1]. Does it follow that product gh of g and h is Riemann integrable on [0, 1]? Justify your answer.

(iii) Suppose that there exists $\sigma \in P[0, 1]$ for each $\varepsilon > 0$ such that $U(f, \sigma) - L(f, \sigma) < \varepsilon$.

Is f Riemann integrable on [0, 1]? Justify your answer.

Give an example of a function f such as above such that $f(1/2) = 2007$.

03.(i) By dissecting interval [1, 2] into n subintervals in geometric progression such that $1 < r < r^2 < \cdots < r^n = 2$ and using the definition of Riemann integral, prove that $\int_1^2 x^4\,dx = \frac{31}{5}$. Hint: $\lim_{x \to 1} \frac{x-1}{x^n - 1} = \frac{1}{n}$.

(ii) Deduce that $\lim_{n \to \infty} \frac{1}{n^5} \sum_{k=1}^{n} (n+k)^4 = \frac{31}{5}$.

(iii) Construct a function f on [1, 2] such that the upper integral of f on [1, 2], $\overline{\int_1^2} f(x)dx = \frac{31}{5}$ and the lower integral of f on [1, 2], $\underline{\int_1^2} f(x)dx = \frac{5}{31}$.

04. (i) Let F be a differentiable function with a Riemann integrable derivative F' on $[a, b]$. Prove that

$$\int_a^x F'(t)dt = F(x) - F(a) \text{ for each } x \in [a, b].$$

(ii) Suppose that f is Riemann intergrable on [0, 1]. Let $F(x) = \int_0^x f(t)dt$ for each $x \in [0, 1]$. Is F differentiable on [0, 1]? Justify your answer.

05. Let n be a positive integer greater than 1.

(i) Prove that if f is continuous and increasing on $[1, \infty)$, then

$$\int_1^n f(x)dx \le \sum_{k=2}^{n} f(k) \le \int_2^{n+1} f(x)dx.$$

(ii) Deduce that

$$n \ln n - n + 1 \le \ln n! \le (n+1)\ln(n+1) - n + 1 - 2\ln 2.$$

(iii) Prove that

$$\frac{n^n}{n!} \le \exp n \le \frac{(n+1)^{n+1}}{n!}.$$

06. Let f be a positive bounded function defined on $[2, \infty)$ such that f is Riemann integrable on $[2, k]$ for each $k > 2$.

(i) Suppose $\lim_{x \to \infty} x^2 f(x) = 0$. Prove that $\int_2^\infty f(x)dx$ converges.

(ii) Does $\lim_{x \to \infty} xf(x) = 0$ imply that f is Riemann integrable on $[2, \infty)$? Justify your answer.

(iii) Find a function f such that $\lim_{x \to \infty} f(x) = 0$ and f is not Riemann integrable on $[2, \infty)$. Justify your answer.