The Open University of Sri Lanka
B.Sc. Degree Programme – Level 04
Final Examination 2006/2007
Pure Mathematics
PMU 2194/PME 4194 – Number Theory & Polynomials

Duration :- Two and Half (2 ½) Hours.

Date :- 22-06-2007. **Time:- 10.00 a.m. – 12.30 p.m.**

Answer Four Questions Only.

01. Let $\mathbb{N}$ be the set of natural numbers. Then prove the followings.

(a) If S is an inductive set and $S \subseteq \mathbb{N}$ then $S = \mathbb{N}$.

(b) If $n \in \mathbb{N}$ then $n \geq 1$.

(c) If $n \in \mathbb{N}$ and $n > 1$, then $n - 1 \in \mathbb{N}$.

(d) If $n \in \mathbb{N}$ there is no $m \in \mathbb{N}$ s. t. $n < m < n + 1$.

(e) If T is a non-empty subset of $\mathbb{N}$, then T has a least element.

(Hint: (i) There is no natural number n such that $0 < n < 1$.
(ii) If $m, n \in \mathbb{N}$ and $m > n$ then $m - n \in \mathbb{N}$.
(iii) If $m, n \in \mathbb{N}$ and $m < n$ then $m + 1 \leq n$.)

02. (a) Let $n_0 \in \mathbb{N}$ and for each natural number $n \geq n_0$, a proposition $p(n)$ satisfies the following conditions:

(i) $p(n_0)$ is true.
(ii) $n \geq n_0$ and $p(n)$ is true $\Rightarrow p(n+1)$ is true.
Prove that $p(n)$ is true for all $n \geq n_0$.

(b) If $b \in \mathbb{Z}$ and $B = \{z \in \mathbb{Z}: z \geq b\}$ then prove that B is a well ordered set.

(c) Find the **greatest common divisor** of 1023 and 453, and express it in the form

$1023m + 453n$ where $m, n \in \mathbb{Z}$.

(d) Find the **least common multiple** of 1023 and 453.

03. (a) Prove that every positive integer $n > 1$ can be expressed uniquely as a product of prime numbers except for the order in which the prime factors occur.

(b) If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then prove that
(i) $a + c \equiv b + d \pmod{m}$
(ii) $a - c \equiv b - d \pmod{m}$
(iii) $a\,c \equiv b\,d \pmod{m}$.

04.(a) Define the greatest common divisor of two polynomials $f(x)$ and $g(x)$.

(b) Let F be a field and $f(x), g(x) \in F[x]\backslash\{0\}$.

If $d = (f, g) \in F[x]$ is the greatest common divisor of f and g then show that d can be expressed in the form $d = fu + gv$ with $u(x), v(x) \in F[x]$ and $d(x)$ is unique. (State clearly any result that you use).

(c) Find the greatest common divisor d of $f = x^{27} - 1$, $g = x^{15} - 1$ In $Z_3[x]$ and express it in the form $d = fu + gv$ with $u, v \in Z_3[x]$.

05.(a) Define an irreducible polynomial.

(b) Let F be a field. Show that $F[x]$ has infinite number of irreducible polynomials.

(c) Show that a quadratic or cubic polynomial $f(x)$ in $F[x]$ is irreducible if $f(\alpha) \neq 0$ for all $\alpha \in F$.

(d) Give a counter example to show that the above result is not true if $\deg f(x) > 3$.

06.(a) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x]$ and $\alpha_1, \ldots, \alpha_n$ are the zeros of f in $\mathbb{C}$, where $\mathbb{C}$ denotes the set of complex numbers. Let s_r be the sum of the products of the α_i's taken r at a time $1 \leq r \leq n$. Show that $s_r = (-1)^r \dfrac{a_{n-r}}{a_n}$.

If α_1, α_2 and α_3 are the zeros of $f(x) = 3x^3 - x + 1$, determine a polynomial $g(x)$ whose roots are $\dfrac{1}{\alpha_1^2}, \dfrac{1}{\alpha_2^2}$ and $\dfrac{1}{\alpha_3^2}$.

(b) Let. $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and $n \geq 1$.

If $\alpha \in \mathbb{Q}$ is a zero of $f(x)$ and $\alpha = \dfrac{r}{s}$ with $(r, s) = 1$, then show that $r \mid a_0$ and $s \mid a_n$.

Find the roots, if any in $\mathbb{Q}$ of $f(x) = 2x^4 + 7x^3 + 3x^2 + 12 = 0$.