The Open University of Sri Lanka
B.Sc. Degree Programme – Level 04
Final Examination 2007/2008
Pure Mathematics
PMU 2194/PME 4194 – Number Theory & Polynomials

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**Duration :- Two and Half (2 ½) Hours.**

**Date :- 01-07-2008.** **Time:- 10.00 a.m. – 12.30 p.m.**

**Answer Four Questions Only.**

01. (a) Show that $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$, where $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of natural numbers.

(b) Let $z_1, z_2 \in \mathbb{Z}$. Prove the following.

(i) $z_1 + z_2 \in \mathbb{Z}$

(ii) $z_1 - z_2 \in \mathbb{Z}$

(iii) $z_1 . z_2 \in \mathbb{Z}$

(c) If $a, b \in \mathbb{Z}$ and $ab = 1$ then prove that $|a| = 1$ and $|b| = 1$.

(Hint: if $n \in \mathbb{N}$ then $n \geq 1$)

02. (a) Let $S = \mathbb{N}$ and $T = \{2n : n \in \mathbb{N}\}$.
Show that there is a bijection between $S$ and $T$.

(b) Let $S$, $T$ and $U$ be any three subsets of the set of integers. Prove that

(i) if $S \sim T$ and $T \sim U$ then $S \sim U$.

(ii) if $S \sim T$ then $T \sim S$.

(iii) if $S \sim T$, then $S$ is finite if and only if $T$ is finite.

(iv) if $S \sim T$, then $S$ is denumerable if and only if $T$ is denumerable.

(c) Prove that $x^n - y^n$ is divisible by $x + y$, when $n$ is an even positive integer.

03. (a) Find the greatest common divisor of 4203 and 207. Express it in the form $4203m + 207n$ with suitable integers $m$ and $n$.
Find the least common multiple of 4081 and 319.

(b) Prove that, if $a, b \in \mathbb{Z}$ with at least one of them non-zero then $a$ and $b$ have a unique greatest common divisor $d$ which can be expressed in the form

$$d = am + bn \qquad \text{with } m, n \in \mathbb{Z}.$$

(Hint: If $S$ is a non-empty subset of $\mathbb{Z}$ such that $S$ is closed under the subtraction, then $S = \{0\}$ or $S$ contains a least positive integer $d$ such that $S = \{nd : n \in \mathbb{Z}\}$.)

(c) Prove that,

(i) if $a + c \equiv b + c \pmod{m}$, then $a \equiv b \pmod{m}$.

(ii) if $ac \equiv bc \pmod{m}$ and $(c, m) = 1$, then $a \equiv b \pmod{m}$

(iii) if $ac \equiv bc \pmod{m}$ and $(c, m) = d$, then $a \equiv b \pmod{m_1}$ where $m = m_1 d$.

04.(a) Define the greatest common divisor of two polynomials.

Let $F$ be a field and let $f, g \in F[x] - \{0\}$. Show that $f$ and $g$ have a unique greatest common divisor $d = (f, g)$ in $F[x]$ and $d$ can be expressed in the form, $d = fu + gv$ with $u, v \in F[x]$.

(b) Find the greatest common divisor $d$ of $f(x) = x^4 + 2x^3 + 3x^2 + 2x + 1$ and $g(x) = x^3 + 3x^2 + 3x + 2$ in $\mathbb{Q}[x]$ and express it in the form $d = fu + gv$ with $u, v \in \mathbb{Q}[x]$ where $\mathbb{Q}$ is the set of rational numbers.

05.(a) (i) Let F be a field and let $f(x)$ be a quadratic or cubic polynomial in $F[x]$. Prove that $f$ is irreducible if $f(\alpha) \neq 0$ for all $\alpha \in F$.

(ii) Discuss the irreducibility of $f(x) = x^3 + x + 1$ in $\mathbb{Z}_5[x]$.

(iii) Give a counter example to show that the above result in not true if deg $f > 3$.

(b) (i) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and $n \geq 1$. Show that if $\alpha \in \mathbb{Q}$ is a zero of $f(x)$ and $\alpha = \frac{r}{s}$ with $(r, s) = 1$, then $r \mid a_0$ and $s \mid a_n$.

(ii) Find the rational roots if any in $\mathbb{Q}$ of $f(x) = 36x^4 - 13x^2 + 1$.

06.(a) Let $f(x) \in \mathbb{R}[x]$ such that deg $f(x) \geq 1$ and let $\alpha \in \mathbb{C}$ such that $f(\alpha) = 0$. Show that $f(\overline{\alpha}) = 0$, where $\overline{\alpha}$ is the conjugate of $\alpha$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.

(b) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]$ be a polynomial of odd degree where $a_n > 0$ and $a_0 \neq 0$. Show that $f$ has a real zero $\alpha$ such that $\alpha a_0 < 0$.

(c) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x]$ be a polynomial of even degree where $a_n > 0$ and $a_0 < 0$. Show that $f(x)$ has two real zeros $\alpha, \beta$ with $\alpha > 0$ and $\beta < 0$.

(d) Show that $1 + i$ is a zero of $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 2$. Also find the other zeros of $f(x)$.