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The Open University of Sri Lanka
B.Sc. Degree Programme – Level 04
Final Examination 2007/2008
Pure Mathematics
PMU 2195/PME 4195 – Theory of Integration

Duration :- One and Half Hours.

Date :- 04–07–2008. **Time:- 10.00 a.m. – 12.30 p.m.**

Answer FOUR Questions Only.

01. Let $[a, b]$ be a closed and bounded interval and let c be a number such that $a < c < b$. Suppose that f is a bounded function on $[a, b]$. Prove that

(i) $\underline{\int_a^c} f(x)dx + \underline{\int_c^b} f(x)dx = \underline{\int_a^b} f(x)dx$

(ii) $\overline{\int_a^c} f(x)dx + \overline{\int_c^b} f(x)dx = \overline{\int_a^b} f(x)dx$

Deduce that if f is Riemann integrable on $[a, b]$, then f is Riemann integrable on both the intervals $[a, c]$, $[c, b]$ and $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$.

02. (i) Let f be an increasing function on the closed and bounded interval $[a, b]$. Prove that f is Riemann integrable on $[a, b]$.

(ii) Let $g(x) = \begin{cases} 1, & x = \frac{1}{2} \\ x, & x \in [0,1] - \left\{\frac{1}{2}\right\}. \end{cases}$

Use the fact that g is increasing on both the intervals $[0, \frac{1}{2} - \delta]$ and $[\frac{1}{2} + \delta, 1]$ for each $0 < \delta < \frac{1}{2}$ to show that g is Riemann integrable on $[0, 1]$.

(iii) Let $h(x) = \begin{cases} 1, & x \in \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\} \\ x, & x \in [0,1] - \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}. \end{cases}$

Prove that h is Riemann integrable on $[0, 1]$.

03. Let (r_n) be an enumeration of the set of all rational numbers in [0, 1] such that $r_1 = 0$ and $r_2 = 1$.

Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = \sum_{r_n \in [x,1]} \frac{1}{3^n}$ for each $x \in [0, 1]$.

(i) Prove that f is bounded and decreasing on [0, 1].

(ii) Use the Riemann Criterion to show that f is Riemann integrable on [0, 1].

(iii) Show that $\frac{1}{9} \le \int_0^1 f(x)dx \le \frac{1}{2}$.

04. (i) Let $f: [a, b] \to \mathbb{R}$ be an integrable function on $[a, b]$ and let $F : [a, b] \to \mathbb{R}$ be defined by $F(x) = \int_x^b f(t)dt$ for each $x \in [a, b]$. Let $c \in [a, b]$ be such that f is continuous at c.

Prove that F is differentiable at c. Find $F'(c)$.

(ii) Let $g : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$.

Define $G : [a, b] \to \mathbb{R}$ by $G(x) = (b-x)\int_a^b g(t)dt - (b-a)\int_x^b g(t)dt$ for each $x \in [a, b]$.

Use Rolle's theorem to prove that there is a point $c \in (a, b)$ such that

$(b-a)g(c) = \int_a^b g(t)dt$.

05. (i) Prove that $\frac{1}{x^2+1} \le \log\left(1+\frac{1}{x^2}\right) \le \frac{1}{x^2}$, for all $x \neq 0$.

(ii) Find an example of a function f defined on $\mathbb{R} - \{0\}$ such that $\lim_{x \to \infty} f(x) = 1$, $\lim_{x \to \infty} [f(x)]^x = 1$ and $\lim_{x \to \infty} [f(x)]^{(x^2)} = 2008$.

06.(i) Which of the following integrals converge? Justify your answer.

(a) $\int_2^\infty \frac{1}{x^2}dx$ (b) $\int_2^\infty \frac{1}{x}dx$ (c) $\int_2^\infty \frac{x}{x^2-1}dx$ (d) $\int_2^\infty \frac{x}{x^3+1}dx$ (e) $\int_0^1 \frac{1}{\sqrt{x}}dx$.

(ii) Is the following correct ? Justify your answer.

$$\int_0^2 \frac{1}{(x-1)^4}dx = \int_0^2 (x-1)^{-4}dx = \left.\frac{(x-1)^{-3}}{-3}\right|_0^2 = \left.\frac{-1}{3(x-1)^3}\right|_0^2 = \frac{-1}{3(2-1)^3} + \frac{1}{3(0-1)^3} = -\frac{2}{3}.$$