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The Open University of Sri Lanka
B.Sc. Degree Programme – Level 04
Final Examination 2008/2009
Pure Mathematics
PMU2193/PME 4193–Real Analysis

Duration: - Two & Half Hours.

Date: - 20-12-2008. **Time: - 1.00 p.m. – 3.30 p.m.**

Answer Four Questions only.

(1) (a) State whether the following propositions are true or false. Justify your answers.

(i) Sum of any two irrational numbers is irrational.
(ii) Sum of any rational number and any irrational number is irrational.
(iii) Between any two irrational numbers, there exists an irrational number.

(b) Guess the limit of each of the following sequences. Use $\varepsilon - N$ definition of a convergent sequence to verify the guesses.

(i) $\left(\dfrac{n+1}{2n+1}\right)$ (ii) $\left(\dfrac{1}{n}+\dfrac{(-1)^n}{n^2}\right)$ (iii) $\left(n(e^{\frac{1}{n}}-1)\right)$.

(2) Let (a_n) be a sequence defined as $a_{n+1} = 1+\dfrac{1}{a_n}$ for all $n \in \mathbb{N}$ with $a_1 = 1$. Show the following.

(i) $a_n - a_{n+2} = \dfrac{a_n^{\,2} - a_n - 1}{a_n + 1}$ for all $n \in \mathbb{N}$.

(ii) The subsequence (a_{2n-1}) is monotonically increasing and bounded above.

(iii) The subsequence (a_{2n}) is monotonically decreasing and bounded below.

(iv) $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = \dfrac{1+\sqrt{5}}{2}$.

(3) Let k be a positive integer.

(i) Show that $\frac{1}{\sqrt{k+1}} \leq \int_k^{k+1} \frac{dx}{\sqrt{x}} \leq \frac{1}{\sqrt{k}}$.

(ii) Set $c_k = \frac{1}{\sqrt{k}} - \int_k^{k+1} \frac{dx}{\sqrt{x}}$. Deduce that $0 \leq \sum_{k=1}^{n} c_k \leq 1 - \frac{1}{\sqrt{n+1}}$ for each positive integer n. Hence show that the sequence $\left(2\sqrt{n} - \sum_{k=1}^{n} \frac{1}{\sqrt{k}}\right)$ converges to some real number in $[1,2]$.

(iii) Show that the integral part of $\sum_{k=1}^{1000000} \frac{1}{\sqrt{k}}$ is equal to 1998.

(4) (a) Prove that $\sum_{n=1}^{\infty} \frac{3n-2}{n(n+1)(n+2)} = 1$. (Hint: evaluate the n^{th} partial sum using partial fractions.)

(b) Determine the convergence or divergence of each of the following series. For each of the convergent alternating series test for absolute or conditional convergence.

(i) $\sum \frac{(-1)^{n+1}}{10}$ (ii) $\sum \frac{n!}{3.5.7...(2n+3)}$

(iii) $\sum \frac{\cos n\pi}{n^2+1}$ (iv) $\sum \frac{(-1)^n}{n+\frac{1}{n}}$.

(5) (a) Use $\varepsilon-\delta$ definition of limit of a function, to show the following.

(i) $\lim_{x\to 0} \cos\frac{1}{x}$ doesn't exist.

(ii) $\lim_{x\to 0} x\sin\frac{1}{x}=0$.

(b) Let $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$

(i) Use the definition of differentiability of a function at a point, to show that $f'(0)$ exists and $f'(0)=0$.

(ii) Deduce that f' is not continuous at 0.

(6) (i) State the Intermediate Value Property of continuous functions.

(ii) Let $f(x)=x^{2008}-12x^{20}-c$ for all $x\in\mathbb{R}$, where c is your OUSL index number. Show that there exists a root of f between 1 and 2.

(iii) Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f(x)$ is rational for all $x\in\mathbb{R}$. Show that f is a constant function.