The Open University of Sri Lanka
Department of Mathematics
B.Sc/ B.Ed Degree Programme
Final Examination - 2017/ 2018
Applied Mathematics– Level 05
APU3244 – Graph Theory

DURATION: - THREE HOURS

Date: 09 – 04 – 2019 **Time: 1.30 p.m. – 4.30 p.m.**

ANSWER FIVE QUESTIONS ONLY

01. Determine whether each of the following statements is true or false: If false, give a counter example and if true, explain.

(a) If sum of the *degrees* of a *regular* graph of *order* 9 is 72, then it has *size* 36 with *degree* of each *vertex* is 8.

(b) Every *regular* graph is a *complete* graph.

(c) *Regular* graph of *order* n with *degree* r has *size* $\frac{nr}{2}$.

(d) All *complete bipartite* graphs are *complete* graphs.

(e) The *size* of a graph will be reduced by the *degree* of the *vertex* v when v is deleted.

(f) The *size* of the complement of the *Petersen graph* is 20.

(g) *Complete* graph of order 4 is a *self-dual* graph.

(h) There is no *self-complementary* simple graph of *order* 5.

(i) There exists a *cube* graph Q_m such that $K_2 \times Q_m = Q_{m+1}$.

(j) There do not exist p and q such that $\overline{K_{p,q}} = K_p \cup K_q$.

02. (a) Define a *Hamiltonian cycle* of a graph G.

(i) For which values of m and n, does the *complete bipartite* graph $K_{m,n}$ have a *Hamiltonian cycle*?,

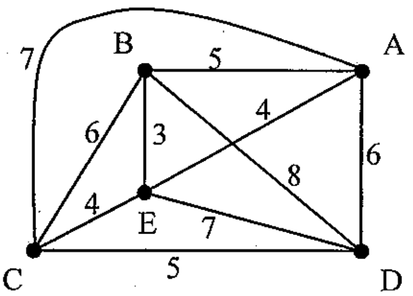
(ii) Give an example that a *connected* graph need not have *Hamiltonian cycle*,

(iii) Do all the *complete* graphs have *Hamiltonian cycles*? Justify your answer,

(iv) Determine the number of *Hamiltonian cycles* in the *complete* graph of *order* 5.

(b) Let A, B and C be three distinct *vertices* of a weighted graph G and let $d(A, B)$ be the *distance* between A and B. Show that any solution to the *travelling salesman problem* for G has weight at least $d(A, B)+d(B, C)+d(C, A)$.

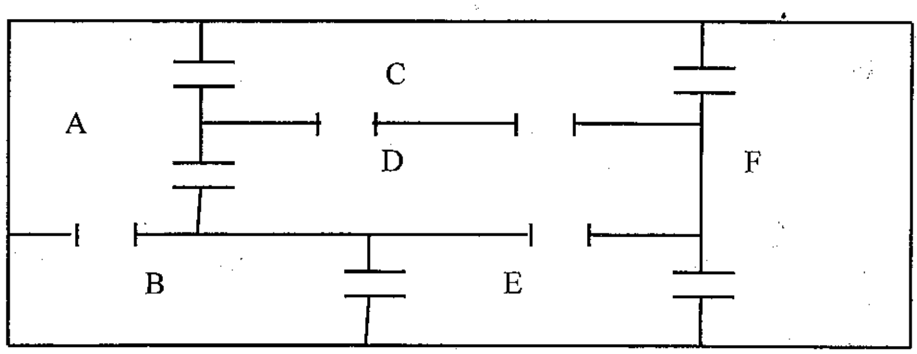
(c) A transporter wants to distribute the products from A, where the manufacturing company is located as shown in the following figure, to all the vendors (B, C, D, E) and come back to A by traveling to each vendor only once. Find the shortest route and its length.



03. If a graph G has an *Euler circuit*, then show that G is *connected* and every *vertex* has even *degree*.

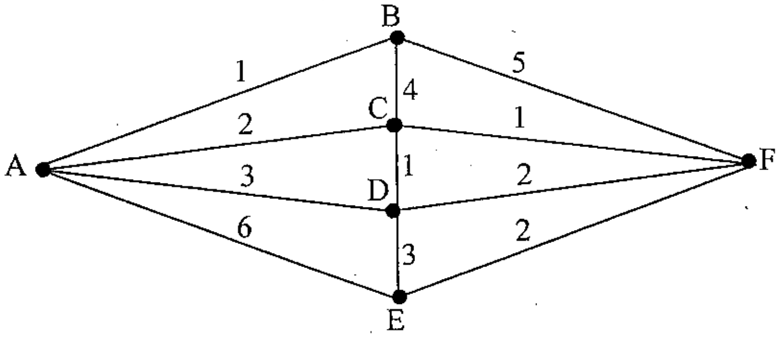
(a) Determine the value of n for which each of the following graphs has an *Euler circuit*:
(i) *Cycle* graph of *order* n (ii) *Complete* graph of *order* n (iii) *Cube* graph of *order* n

(b) Is it possible for someone to walk through each gate exactly once and returns to its starting point A as shown in the following map of the floor plan of a factory? Justify your answer.



Find a path from the room A to E such that it passes through each gate once and only once.

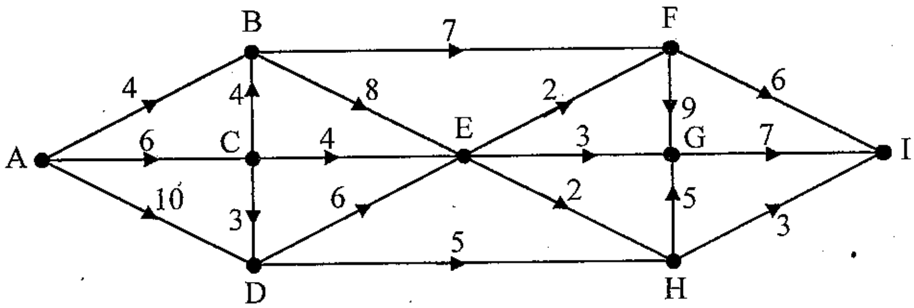
(c) A mail carrier has to deliver mail to all streets in a sector as shown in the following figure. The weight of each edge represents the length of each street.



(i) Suggest two locations which are most suitable to start the delivery in a way to visit all the streets only once. Give reasons for your suggestion,

(ii) If the mail carrier has to start at one of these locations and end at the same location with minimum length of traversed, what is the most efficient route for the mail carrier? Hence, compute the length of the shortest route.

04. Identify the *source* and the *sink* of the following *network* N.



(a) What is the total weight of the *critical path* of N?

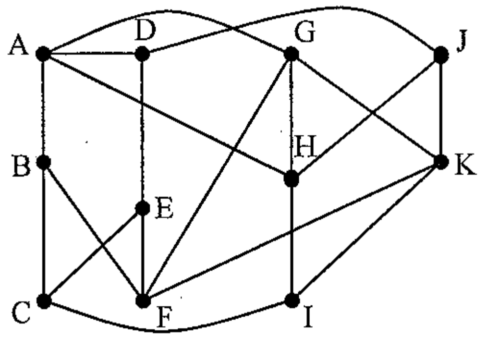
(b) Write down all the *edge-disjoint paths* and *vertex- disjoint paths* from A to I in N.

(c) Find a *minimal* AI- *disconnecting* set and a *minimal* AI- *separating set* in N. Hence, verify the *Menger's theorems* for N.

(d) Draw the *maximum flow* of N. Hence, verify the *maximum flow - minimum cut theorem* for N.

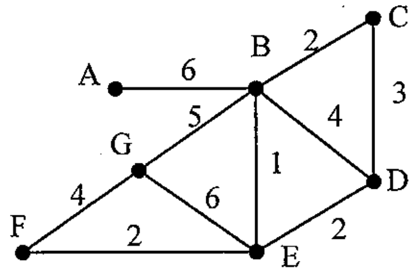
05. Let T be a *tree* with n *vertices*. Prove that T contains no *cycles* and has $n-1$ number of *edges*.

(a) Use the *Depth First Search* algorithm to produce a spanning tree of maximum height, by choosing 'A' as the root.



Find a spanning tree of minimum height of the above graph by using the *Breadth First Search* algorithm.

(b) Use *Kruskal's Greedy* algorithm to find the minimum weighted *spanning tree* for the following weighted graph:



Verify the above result using *Prim's Greedy* algorithm, by starting with the edge AB.

06. Let $G(V_1, V_2)$ be a *bipartite* graph. Define a *complete matching* from V_1 to V_2.

(a) Suppose that 4 boys *A, B, C* and *D* know 5 prospective girls *P, Q, R, S* and *T* as given in the following table:

Boy	Girls known by a boy
A	*Q* *S*
B	*Q* *S*
C	*P* *R* *T*
D	*Q* *S*

(i) Draw the *bipartite* graph according to the above relationships,

(ii) Determine whether the graph obtained in part (i) has a *complete matching*,

(iii) Is *marriage condition* satisfied for this problem? Justify your answer.

(b) Find the *term rank* of the following *incidence matrix* A of a *family* $\Im$ of non-empty subsets of a finite set

$$A=\begin{bmatrix}0&1&0&1&0&0\\0&0&1&1&0&0\\1&0&1&0&1&1\\0&1&0&1&0&0\\0&0&1&1&0&0\end{bmatrix}.$$

Hence,

(i) Verify the *Konig-Egervary* theorem for A.

(ii) Is $\Im$ has *transversal*? Justify your answer.

List 3 distinct *partial transversals* of $\Im$ with 4 sets.

07. Define $k-colorable(v)$ and $k'-colorable(f)$ of a graph G, where v is any *vertex* of G and f is any *face* of G.

(a) Find k and k' of each of the following graphs:

(i) *Complete* graph of *order* 4 (ii) *Wheel* graph of *order* 5 (iii) *Cube* graph of *order* 3

Which of those are *color critical*? Justify your answer.

(b) Suppose that a chemist wishes to store five chemicals *A*, *B*, *C*, *D* and *E* in various areas of a warehouse. Some of these chemicals react violently when in contact, and so must be kept in separate areas. In the following table, an asterisk indicates those pairs of chemicals that must be separated.

	A	*B*	*C*	*D*	*E*
A	--	*	*	*	--
B	*	--	*	*	*
C	*	*	--	*	--
D	*	*	*	--	*
E	--	*	--	*	--

(i) Draw a graph G whose five *vertices* correspond to the five chemicals with two *vertices* are *adjacent* whenever the corresponding chemicals are to be kept separately,

(ii) Find the values of k and k' of the graph G,

(iii) Draw the *dual* graph G^* of G.

Is there any relationship between the value k' of G and the value k of G^*? Justify your answer.

08. (a) Let $M = (E, I)$ be a *matroid* defined in terms of its independent sets. Let A be a subset of a non-empty finite set E. Define the *rank of* A.

Let $E = \{a, b, c, d, e\}$ be a set of five elements. Find *matroids* on E and write down their standard names in each of the following cases:

(i) E is the only base,

(ii) the empty set is the only base,

(iii) the bases are the subsets of E containing exactly three elements.

(b) Let $E = \{a, b, c, d, e, f, g\}$ be a set of seven elements.

Let $B = \{bcd, bef, bga, ceg, cfa, dea, dfg\}$ be a *family* of 3-element subsets of E.

Draw a *Fano matroid* defined on the set E.