The Open University of Sri Lanka
B.Sc. Degree Programme – Level 05
Open Book Test (OBT) 2008/2009
PMU 3292/PME 5292 – Group Theory and Transformation
Pure Mathematics

Duration :- One and Half Hours.

Date :- 29. 09. 2008. **Time:- 4.00 p.m. – 5.30 p.m.**

Answer All Questions.

01.(a) Decide whether each of the following relations on the set given are reflexive (R); symmetric (S); transitive (T). If you decide a relation does not have one of the properties, give a counter example.

(i) On the set $\mathbb{Z}^+$; $a\,R\,b$ means a is a multiple of b.

(ii) On the set $\mathbb{Z}$; $a\,R\,b$ means ab is a multiple of 5.

(iii) On the set $\mathbb{R}$; $x\,R\,y$ means $x \neq y$

(iv) On the set $\mathbb{Z}$; $a\,R\,b$ means $a - b$ is divisible by 2.

(b) Decide whether each of the following mappings is injective, surjective or bijective.

(i) $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by $\alpha(n) = n + 3$.

(ii) $\alpha : \mathbb{Z} \to \mathbb{Z}$ defined by $\alpha(r) = -r$.

(iii) $\alpha : \mathbb{Q} \to \mathbb{Q}$ defined by $\alpha(q) = 3q - 1$.

(iv) $\alpha : \mathbb{Z} \to \mathbb{Z}$ defined by $\alpha(x) = x^2$.

02. Check whether the following systems form a group or not.

(a) G = set of rational numbers under composition * defined by $a * b = \frac{ab}{2}$, $a, b \in G$.

(b) $G = \{1, w, w^2\}$ where w is a cube root of unity under multiplication.

(c) $M = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \middle| \, a, b \text{ reals, } a + b \neq 0 \right\}$ under matrix multiplication.

03. Let G denote the set of all ordered pairs $(a,\ b)$ where a is a non–zero real numbers and $b = \pm 1$. For $(a_1, b_1), (a_2, b_2) \in G$, set $(a_1, b_1) \odot (a_2, b_2) = (a_1 a_2^{b_1}, b_1 b_2)$.

(a) Compute $(2, -1) \odot (3, 1)$, $(2, -1) \odot (2, -1)$ and $(2, 1) \odot (\frac{1}{2}, 1)$.

(b) Show that G is a group under $\odot$.

(c) Is G abelian ?

(d) Which elements of G are their own inverses?

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Model Answers.

(01) (a) (i) R, T, not S eg. 6 R 2 but 2 $\not R$ 6

(ii) S, not R eg. 1 $\not R$ 1
not T eg. 2 R 5 and 5 R 1
but 2 $\not R$ 1

(iii) S, not R any $n \in R$
not T eg. 1 R 2 and 2 R 1
but 1 $\not R$ 1

(iv) R, S, T.

(b) (i) Injective
(ii) injective, Surjective
(iii) Injective, Surjective
(iv) NO.

(02) (a) $G = (\mathbb{R}, *)$, $*$ defined by $a * b = ab/2$
$a, b \in G$. clearly G is closed.
for any $a, b, c \in \mathbb{R}$
$a * (b * c) = a * bc/2 = abc/4$
$(a * b) * c = ab/2 * c = abc/4$
$\therefore G$ is associative.

for any a, G, $a * e = e * a = a$
$a * e = ae/2 = a = ea/2 = e * a$
$e = 2$
for any $a \in G$, $\exists\, b \in G$ such that
$ab = ba = 2$

(b) $G = \{1, \omega, \omega^2\}$, $\omega^3 = 1$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

G is closed, associative, 1 is the identity, inverses of $1, \omega, \omega^2$ are $1, \omega^2, \omega$ respectively. $\therefore G$ is a group.

(c) $M = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} \middle| a, b \text{ reals, } a+b \neq 0 \right\}$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \neq 0$$

$$\begin{pmatrix} a_1 & a_1 \\ b_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & a_2 \\ b_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + a_1 b_2 & a_1a_2 + a_1b_2 \\ b_1a_2 + b_1b_2 & b_1a_2 + b_1b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_1 \\ b_1 & b_2 \end{pmatrix} (a_2 + b_2) \in M$$

$$a_2 + b_2 \neq 0$$

M is closed

matrix multiplication is associative.

Let $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$ be the identity of $\begin{pmatrix} c & c \\ d & d \end{pmatrix}$

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix}\begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} c & c \\ d & d \end{pmatrix}\begin{pmatrix} a & a \\ b & b \end{pmatrix} = \begin{pmatrix} c & c \\ d & d \end{pmatrix}$$

$$(a+b)\begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} c & c \\ d & d \end{pmatrix}$$

$a + b = 1$, $a = 1/2$, $b = 1/2$ or

$$\begin{pmatrix} ac+ad & ac+ad \\ bc+bd & bc+bd \end{pmatrix} = \begin{pmatrix} c & c \\ d & d \end{pmatrix}$$

$$(c+d)\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} c & c \\ d & d \end{pmatrix}$$

$(c+d) a = c \qquad a = c/_{c+d}$

$(c+d) b = d \qquad b = d/_{c+d}$

$c+d \neq 0$.

$\therefore$ identity element is not unique. $\therefore$ M is not a group.

(03) (a) $(2,-1) \odot (3,1) = (2/3, -1)$

$(2,-1) \odot (2,-1) = (1, 1)$

$(2, 1) \odot (1/2, 1) = (1, 1)$

(b) $G = \{(a, b) \mid a \in \mathbb{R}^*, b = \pm 1\}$

clearly G is non empty and closed.

Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in G$

$\{(a_1, b_1) \odot (a_2, b_2)\} \odot (a_3, b_3) = (a_1 a_2^{b_1} a_3^{b_1 b_2}, b_1 b_2 b_3)$

$(a_1, b_1) \odot \{(a_2, b_2) \odot (a_3, b_3)\} = (a_1 a_2^{b_1} a_3^{b_2 b_1}, b_1 b_2 b_3)$

$\therefore$ G is associative.

Suppose for any $(a, b) \in G$, $(x, y) \in G$ is the identity of G.

$(a, b) \odot (x, y) = (x, y) \odot (a, b) = (a, b)$

$(a, b) \odot (x, y) = (a, b)$

$(ax^b, by) = (a, b)$

$ax^b = a, \quad by = b$

$y = 1 \quad (b \neq 0)$

$x^b = 1 \quad (a \neq 0)$

$\therefore x = 1$

$(x, y) \odot (a, b) = (xa^y, yb) = (a, b)$

$xa^y = a, \quad yb = b \quad (b \neq 0)$

$y = 1, \quad \therefore xa = a, \quad x = 1$
$(a \neq 0)$

$\therefore$ $(1, 1)$ is the identity element of G.

Suppose $(a, b) \in G$, let (x, y) be the inverse of (a, b).

Then $(a, b) \odot (x, y) = (x, y) \odot (a, b) = (1, 1)$

$(a, b) \odot (x, y) = (ax^b, by) = (1, 1)$

$ax^b = 1, \quad by = 1$

$y = 1/b, \quad b \neq 0.$

$x^b = 1/a, \quad x = (a^{-1})^{1/b}$

$(x, y) \odot (a, b) = (1, 1)$

$(xa^y, yb) = (1, 1).$

$xa^y = 1, \quad yb = 1, \quad y = 1/b$

$xa^{1/b} = 1, \quad x = (a^{-1})^{1/b}$

$\therefore$ $(a^{-1/b}, 1/b)$ is the inverse of

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Duration :- One and Half Hours.

Date :- 27 - 10 - 2008. **Time:- 4.00 p.m. – 5.30 p.m.**

Answer All Questions.

01. (a) Show that the set $H = \{2^n : n \in Z\}$ of all powers of 2 is a subgroup of $(\mathbb{R}\setminus\{0\}, \cdot)$.

(b) Let $H = \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}\right\}$, the set of 2×2 matrices representing shears parallel to the x-axis. show that H is a subgroup of $GL_2(\mathbb{R})$. Is H abelian?

02. (a) Find the order of each of the elements 2, 3 in the group $(Z_7^*, \otimes)$.

(b) Find the order of each of the elements 2, 3, 4 in the group $(Z_8, \oplus)$.

(c) Find all the generators of $(Z_{10}, \oplus)$.

03. (a) Show that the product $(1\ 6)(1\ 5)(1\ 4)(1\ 3)(1\ 2)$ can be expressed as a single cycle.

(b) Find the order of the following permutations in S_9.

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 5 & 6 & 4 & 3 & 2 & 1 \end{pmatrix}$.

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 7 & 9 & 5 & 8 & 1 & 6 & 4 \end{pmatrix}$.

(iii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 9 & 6 \end{pmatrix}$.

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Closed Book Test Model answers.

(01) (a) $H = \{2^n, n \in \mathbb{Z}\}$

$2^0 = 1, \in H \quad \therefore H \neq \emptyset.$

$2^n, 2^m \in H \quad , m, n \in \mathbb{Z}$

$2^n \cdot 2^m = 2^{n+m} \in H \quad , n+m \in \mathbb{Z}$

H is closed.

let $2^m \in H \quad , 2^{-m} \in H \quad -m \in \mathbb{Z}.$

$2^{m+(-m)} = 2^0 = 1 \quad , 0 \in \mathbb{Z}.$

$\therefore 2^{-m}$ is the inverse of 2^m.

$\therefore$ H is a subgroup of $\mathbb{R} \setminus \{0\}$.

(b) $H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in \mathbb{R} \right\}.$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in H, \quad H \neq \emptyset.$

let $\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \in H \quad , a_1, a_2 \in \mathbb{R}$

$\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1+a_2 \\ 0 & 1 \end{pmatrix} \in H$

$a_1 + a_2 \in \mathbb{R}.$

$\therefore$ H is closed.

Inverse of $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \in H$

$-a \in \mathbb{R}.$

$\therefore$ H is a subgroup of $Gl_2(\mathbb{R})$

$\begin{pmatrix} 1 & a_1 \\ & \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ & \end{pmatrix} = \begin{pmatrix} 1 & a_1+a_2 \\ & \end{pmatrix} = \begin{pmatrix} 1 & a_2 \\ & \end{pmatrix} \begin{pmatrix} 1 & a_1 \\ & \end{pmatrix}$

(Q2) (a) $\mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$

$a = 2,\quad a^2 = 4,\quad a^3 = a^2 \otimes a = 4 \otimes 2 = 1$

$O(2) = 3$

$a = 3 \quad a^2 = 3 \otimes 3 = 2,\quad a^3 = a^2 \otimes a = 6$

$a^4 = a^3 \otimes a = 4$

$a^5 = a^4 \otimes a = 5,\quad a^6 = a^5 \otimes a = 1$

$O(3) = 6.$

(b) $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

identity is 0.

$a = 2 \quad 2a = 2 + 2 = 4$

$3a = 2a + a = 6$

$4a = 3a + a = 0$

$O(2) = 4$

$O(3) = 8,\quad O(4) = 2.$

(c) $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

r	0	1	2	3	4	5	6	7	8	9	
r(1)	0	1	2	3	4	5	6	7	8	9	
r(2)	0	2	4	6	8	0					
r(3)	0	3	6	9	2	5	8	1	4	7	0

$n = 10$, ~~r(1)~~ 1 is a generator if and only if $\gcd(r, n) = 1$

$2(1) = 2,\quad 4(1) = 4,\quad 6(1) = 6,\quad 8(1) = 8$

$3(1) = 3,\quad 5(1) = 5 \quad 7(1) = 7,\quad 9(1) = 9$

$(2, 10) \neq 1,\quad (4, 10) \neq 1 \quad (6, 10) \neq 1$

(09) (a) $(1\ 2\ 3\ 4\ 5\ 6)$

(b) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 5 & 6 & 4 & 3 & 2 & 1 \end{pmatrix}$

$= (1\ 9)(2\ 8)(3\ 7)(4\ 5\ 6)$

$O(\alpha) = \text{L.c.d}\ (2, 2, 2, 3)$

$= 6$

$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 7 & 9 & 5 & 8 & 1 & 6 & 4 \end{pmatrix}$

$= (1\ 2\ 3\ 7)(4\ 9)(6\ 8)$

$O(\beta) = \text{l.c.}(4, 2, 2)$

$= 4$

$\gamma = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9)$

$O(\gamma) = \text{l.c.d.}\ (5, 4)$

$= 20.$