The Open University of Sri Lanka
B.Sc. Degree Programme – Level 04
Final Examination 2008/2009
Pure Mathematics
PMU 2191/PME 4191– Vector Analysis.

101

Duration :- Two and Half Hours.

Date :- 03–01–2009. **Time:- 1.00 p.m. – 3.30 p.m.**

Answer Four Questions Only.

01.(a) (i) Show that $u(x,y)=\tan^{-1}(y/x)$ and $v(x,y)=\tan^{-1}(x/y)$ both satisfy the equation of the form $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=0$.

(ii) Hence show that $z(x,y)=x^2u-y^2v$ satisfies the equation $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$.

(iii) Show also that $\frac{\partial^2 z}{\partial x^2}=2u+2x\frac{\partial u}{\partial x}$ and $\frac{\partial^2 z}{\partial y^2}=-2v-2y\frac{\partial v}{\partial y}$.

(b) Find the second order Taylor polynomial for the function $f(x, y) = x^2y + \sin xy$ about the point $\left(1, \frac{\pi}{2}\right)$.

02. (a) Find the stationary points of the function $f(x,y)=2x^3+y^3+3x^2-18y^2+81y+5$ and determine their nature.

(b) Find the directional derivative of $f(x, y, z) = xy + z^2 + 1$ at the point (1, 3, 2) in the direction of $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$.

(c) Let $\underline{r}=x\underline{i}+y\underline{j}+z\underline{k}$ and $\underline{a}$ is a constant vector show that,

(i) grad $\frac{1}{r}=-\frac{\underline{r}}{r^3}$, (ii) grad $(\underline{a}\cdot\underline{r})=\underline{a}$,

(iii) curl $(\underline{a}\times\underline{r})=2\underline{a}$, where $r=\sqrt{x^2+y^2+z^2}$.

03.(a) Prove that gradient vector of a scalar field $\phi(x, y, z)$ is normal to the contour surface $\phi(x, y, z)=c$ where c is a constant.

(b) Show that the equation of the tangent plane and normal line to the surface $z = f(x, y)$ at the point $P_0(x_0, y_0, z_0)$ are given by

$$z-z_0=\left(\frac{\partial f}{\partial x}\right)_{P_0}(x-x_0)+\left(\frac{\partial f}{\partial y}\right)_{P_0}(y-y_0) \text{ and } \frac{x-x_0}{\left(\frac{\partial f}{\partial x}\right)_{P_0}}=\frac{y-y_0}{\left(\frac{\partial f}{\partial y}\right)_{P_0}}=\frac{z-z_0}{-1}.$$

(b) Find the equation of the tangent plane and normal line to the surface S in $\mathbb{R}^3$ given by $z=x^2y^3-y+1$ at the point (1, 1, 1).

04. (a) Let A be the closed planar region bounded by curves $y = x$, $2y = x$, $xy = 1$ and $xy = 2$ in the first quadrant. Evaluate the surface integral of the function $f(x, y) = \frac{x}{y}$ over the region A.

(b) Find $\int_B z dv$, where B is the region given by $x^2 + y^2 \leq a^2$ and $0 \leq z \leq h$.

(c) Evaluate $\int_B (x^2 + y^2) dv$, where B is the region bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and the planes $z = 0$, $z = 1, y = 0$ and $y = x$, in the first octant.

(d) Find the volume of the region bounded by $x^2 + y^2 + z^2 \leq 9$ and $2 \leq z \leq 3$.

05. (a) State the Gauss divergence theorem.

(b) Consider the vector field $\underline{F} = (3x^3 - x^2) y\underline{i} + (y^3 - 2y^2 + y) x\underline{i} + (z^2 - 1)\underline{k}$ and let S be the surface of a unit cube with one corner at (0, 0, 0), another corner at (1, 1, 1) and aligned with edges along the x, y and z axes. Use divergence theorem to evaluate

$$I = \oint_S \underline{F} \cdot \underline{n} \, dA.$$

Verify your result by calculating the integral directly.

06. (a) (i) Define what is meant by a conservative vector field.

(ii) Given a vector field $\underline{A} = A_1(x, y)\underline{i} + A_2(x, y)\underline{j}$ and a function $\varphi(x, y)$ defined in $\mathbb{R}^2$, show that if $\varphi\underline{A}$ is a conservative vector field then $\varphi\left(\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x}\right) = A_2 \frac{\partial \varphi}{\partial x} - A_1 \frac{\partial \varphi}{\partial y}$.

(b) (i) Given two function $P(x, y)$ and $Q(x, y)$ defined in $\mathbb{R}^2$, prove green's theorem, $\oint_C (Pdx + Qdy) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$, where c is a simple closed curve bounding a region S in $\mathbb{R}^2$.

(ii) Through an appropriate choice for P and Q in (i) above, find an expression for the area of the region S, and apply this to evaluate the area of the ellipse bounded by the curve

$x = a \cos \theta$, $y = b \sin \theta$, where $0 \leq \theta \leq 2\pi$.