The Open University of Sri Lanka
B.Sc. Degree Programme – Level 04
Final Exam 2008/2009
Pure Mathematics
PMU 2195/PME 4195 – Theory of Integration

Duration :- Two and Half Hours.

Date :- 16.07.2009 **Time:- 10.00 a.m. – 12.30 p.m.**

Answer Four Questions Only.

01. Let $f(x)=\begin{cases}0, & x\in\mathbb{Q}\cap[0,1]\\ 1, & x\in(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]\end{cases}$ and $g(x)=\begin{cases}2, & x\in\mathbb{Q}\cap[0,1]\\ 1, & x\in(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]\end{cases}$.

(i) Find the lower integrals;

$$\underline{\int_0^1} f(x)\,dx,\ \underline{\int_0^1} g(x)\,dx,\ \underline{\int_0^1}\left[f(x)+g(x)\right]dx.$$

(ii) Find the upper integrals;

$$\overline{\int_0^1} f(x)\,dx,\ \overline{\int_0^1} g(x)\,dx,\ \overline{\int_0^1}\left[f(x)+g(x)\right]dx.$$

(iii) What can be said about the integrability of each of the functions f, g and $f+g$ on [0,1] ? Justify your answer.

02. (i) Let $f(x)=\begin{cases}0, & x=0\\ 1, & 0<x<1.\\ 2, & x=1\end{cases}$

Use Riemann criterion to show that f is Riemann integrable on [0,1].

(ii) By dividing [1,2] into n subintervals in geometric progression at the points $1, r, r^2, \ldots, r^{n-1}, 2$, prove that $\frac{1}{2r}\le \underline{\int_1^2}\frac{1}{x^2}dx \le \overline{\int_1^2}\frac{1}{x^2}dx \le \frac{r}{2}$.

Deduce that $\int_1^2 \frac{1}{x^2}dx=\frac{1}{2}$.

03. Find each of the following limits. Show your work and state the results you use.

(i) $\lim_{n\to\infty}\frac{1}{n^6}\sum_{k=1}^{n}(n+k)^5$ (ii) $\lim_{n\to\infty} n^2\sum_{k=1}^{n}\frac{1}{(n+k)^3}$.

04.(i) Let f be a Riemann integrable function on $[0,1]$ and let,

$F(x) = \int_0^x f(t)dt$, for each $x \in [0,1]$. Suppose f is continuous at a point $c \in (0,1)$.

Prove that F is differentiable at c and $F'(c) = f(c)$.

(ii) Let g be a bounded function on $[0,1]$. Let $\overline{G}(x) = \overline{\int_0^x} g(t)dt$, the upper integral of g on $[0, x]$ for each $x \in (0,1]$ and $\overline{G}(0) = 0$. Assume that $\overline{G}$ is differentiable on $[0,1]$ and $\overline{G}'(x) = 1$ for each $x \in [0,1]$. Does it follow that g is continuous on $[0,1]$? Justify your answer.

05. (i) Prove the following inequalities.

(a) $1 + x \geq e^{\frac{x}{1+x}}$ for each $x \geq 0$

(b) $e^{\frac{x}{1-x}} \geq \frac{1}{1-x} \geq e^x$ for each x such that $1 > x > 0$.

(ii) Find $\lim_{x \to \infty} \frac{x^n}{e^{\frac{x}{n}}}$, where $n = 10^{133}$, stating clearly the results you use.

(iii) Give an example of a sequence (a_n) of positive real numbers such that for each $k \in \mathbb{N}$, $\lim_{n \to \infty} a_n{}^k = 1$, $\lim_{n \to \infty} a_n{}^{\sqrt{n}} = 1$, and $\lim_{n \to \infty} a_n{}^n = 2009$. Justify your answer.

06. (i) Show that each of the following integrals converge and evaluate each one of them.

(a) $\int_{-1}^{1} \frac{1}{x^{\frac{1}{3}}} dx$ (b) $\int_{-\infty}^{0} xe^x\, dx$

(ii) Discuss the convergence of each of the following integrals.

(a) $\int_{-1}^{1} \frac{1}{x} dx$ (b) $\int_{2}^{\infty} \frac{x^2+1}{x^3+2} dx$

(iii) Is the following argument correct? Justify your answer.

$$\begin{aligned}
\int_{-\infty}^{\infty} (x^3 - x)dx &= \lim_{R \to \infty} \int_{-R}^{R} (x^3 - x)dx \\
&= \lim_{R \to \infty} \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-R}^{R} \\
&= \lim_{R \to \infty} \left(\frac{R^4}{4} - \frac{R^2}{2} \right) - \left(\frac{(-R)^4}{4} - \frac{(-R)^2}{2} \right) \\
&= \lim_{R \to \infty} 0 \\
&= 0.
\end{aligned}$$