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PMU 3292/PME 5292
MODEL ANSWER
OBT/CBT
LEVEL 05
B.Sc.

THE OPEN UNIVERSITY OF SRI LANKA

B.Sc. DEGREE PROGRAMME

PURE MATHEMATICS

2009/2010

LEVEL 05

MODEL ANSWERS – OBT/CBT

PMU 3292/PME 5292 - GROUP THEORY AND TRANSFORMATION

(English)

2010

The Open University of Sri Lanka
B.Sc. Degree Programme-Level 05
Open Book Test (OBT) 2009/2010
PMU3292/PME5292-Group Theory and Transformation
Pure Mathematics
Model Answers

01. $(a,b)\circ(c,d)=(a+c,2^{-c}b+d)$ clearly G is closed.

$$(a,b),(c,d),(e,f)\in G$$

$$(a,b)\circ[(c,d)\circ(e,f)]=(a,b)\circ[c+e,2^{-e}d+f]$$
$$=(a+c+e,2^{-c-e}b+2^{-e}d+f)$$

For any $[(a,b)\circ(c,d)]\circ(e,f)=(a+c,2^{-c}b+d)\circ(e,f)$

$$=(a+c+e,2^{-e}(2^{-c}b+d)+f)$$
$$=(a+c+e,2^{-e-c}b+2^{-e}d+f)$$

G is associative.

If $(c,d)\in G$ is the identity element

$$(a,b)\circ(c,d)=(c,d)\circ(a,b)\quad=(a,b)$$
$$=(a+c,2^{-c}b+d)=(a,b)$$
$$a+c=a$$
$$c=a$$
$$2^{-c}b+d=b$$
$$b+d=b$$
$$d=0$$

$$(c,d)\circ(a,b)=(c+a,2^{-a}d+b)=(a,b)$$
$$a+c=a$$
$$c=0$$
$$2^{-a}d+b=b$$
$$2^{-a}d=0$$
$$2^{-a}\neq 0$$
$$d=0$$

$(0,0)$ is the identity element.

Let (c,d) be the inverse of (a,b).

$$(a,b)\circ(c,d)=(c,d)\circ(a,b)=(0,0)$$

$$(a,b)\circ(c,d)=\left(a+c,2^{-c}b+d\right)=(0,0)$$

$$a+c=0$$

$$c=-a$$

$$2^{-c}b+d=0$$

$$-2^{a}b=d$$

$$(c,d)\circ(a,b)=\left(c+a,2^{-a}d+b\right)=(0,0)$$

$$c+a=0$$

$$c=-a$$

$$2^{-c}d+b=0$$

$$-2^{a}b=d$$

Inverse of (a,b) is $\left(-a,-2^{a}b\right)\in\mathbb{Z}\times\mathbb{Q}$. Therefore every element has an inverse element.

$\therefore$ G is a group under the given binary operation.

$(a,b)\cdot(c,d)=\left(a+c,2^{c}b-d\right)$. Clearly G is closed.

$$(a,b),(c,d),(e,f)\in G$$

$$\left[(a,b)\cdot(c,d)\right]\cdot(e,f)=\left(a+c,2^{c}b-d\right)\cdot(e,f)$$

$$=\left(a+c+e,2^{e}\left(2^{c}b-d\right)-f\right)$$

$$=\left(a+c+e,2^{e+c}b-2^{e}d-f\right)$$

$$(a,b)\cdot\left[(c,d)\cdot(e,f)\right]=(a,b)\cdot\left(c+e,2^{e}d-f\right)$$

$$=\left(a+c+e,2^{e+c}b-2^{e}d+f\right)$$

Since L.H.S $\neq$ R.H.S, G is not associative. Therefore G is not a group.

02. (a)

$$W_1=\left\{\begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix}, x\in\mathbb{Q}^*\right\}$$

$$1\in\mathbb{Q}^*, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\in W_1,$$

W_1 is not empty. Let

$$\begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} y & 2y \\ 0 & 0 \end{pmatrix} \in W_1, x, y \in \mathbb{Q}^*$$

$$\begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y & 2y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} xy & 2xy \\ 0 & 0 \end{pmatrix} \in W_1$$

Where $xy \in \mathbb{Q}^*$. $\therefore W_1$ is closed.
Matrix multiplication is associative.
Let $\begin{pmatrix} y & 2y \\ 0 & 0 \end{pmatrix}$ be the identity element.

$$\begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y & 2y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y & 2y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xy & 2xy \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix}$$

$$xy = x$$

$$y = 1$$

$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ is the identity.

Let $\begin{pmatrix} y & 2y \\ 0 & 0 \end{pmatrix}$ be the inverse of $\begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix}$.

$$\begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y & 2y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y & 2y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 2x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} xy & 2xy \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

$xy = 1$

$y = \frac{1}{x} \in \mathbb{Q}^*$, $\begin{pmatrix} \frac{1}{x} & \frac{2}{x} \\ 0 & 0 \end{pmatrix}$ is the inverse.

$\therefore W_1$ is a group.
(b) The same as above, W_2 is also a group.

03. (a) $Z_{10} = \{0,1,2,3,4,.....,9\}$. Let $S = \{1,3,7,9\}$

×	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

1 is the identity. Inverse of 1,3,7,9 are 1,7,3,9 respectively.

(b) $P = \{1,9\}$

×	1	9
1	1	9
9	9	1

© $Z_7 = \{0,1,......6\}$, $T = \{1,2,4\}$

×	1	2	4
1	1	2	4
2	2	4	1
4	4	1	2

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(01). (a) (i) In the additive group of rationals the binary operation is the usual addition. Then 1^3 means $1\circ 1\circ 1$ where $\circ$ is the binary operation in $\mathbb{Q}$. Hence $1^3=1+1+1=3$. Also 1^{-4} means $\left(1^{-1}\right)^4$ i.e. $1^{-1}\circ 1^{-1}\circ 1^{-1}\circ 1^{-1}$ where $\circ$ is the binary operation under discussion. Now $1^{-1}=-1\in(\mathbb{Q},+)$. Thus $1^{-4}=(-1)+(-1)+(-1)+(-1)=-4$.

(ii)

$$2^2=2.2=4$$

$$2^{-3}=\left(2^{-1}\right)^3=\left(\tfrac{1}{2}\right)^3=\tfrac{1}{8}$$

(iii) $\sigma^2=I, \sigma^3=I\sigma=\sigma=\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$.

(b) (i) 2 (ii) 4

(02) (a) $H=\{2,4,6,8\}\subseteq\mathbb{Z}_{10}$

$H=\left\{a^n, n\in\mathbb{Z}, a\in H\right\}$

	1	2	3	4	5	6
2	2	4	8	6	2	
4	4	6	4			
6	6	6				
8	8	4	2	6	8	

$\langle 2\rangle=H$

$\langle 8\rangle=H$

$\langle 2\rangle$ and $\langle 8\rangle$ are generators of H.

(b) Let G be abelian and H any subgroup of G. Then if $g \in G$ and $h \in H, g^{-1}hg = h$; for since G is abelian, $gh = hg$ and hence multiplying by g^{-1} on the left, $h = g^{-1}hg$. Then if $h \in H, g^{-1}hg \in H$. Thus H is a normal subgroup of G.

(c) If $G = \{1\}$, there is nothing to prove. If $1 \neq g \in G, \langle g \rangle = H$ is a subgroup of G. Hence its order, by Lagrange's theorem, divides the prime $|G|$. As $|H| \neq 1, |H| = |G|$ since the only divisors of $|G|$ are 1 and $|G|$. Thus $H = G$, as $H \subseteq G$ and H and G have the same number of elements.

(03)(i) θ is certainly a mapping of $\mathbb{R}^+$ onto $\mathbb{R}$. Furthermore,

$$(xy)\theta = \log_{10}(xy) = \log_{10}(x) + \log_{10}(y)$$
$$= x\theta + y\theta$$

$\therefore \theta$ is a homomorphism.

$$Ker\,\theta = \{x \mid \log_{10} x = 0, x \in \mathbb{R}^+\}$$
$$= \{1\}$$

We assert that $\mathbb{R}^+\theta = \mathbb{R}$. To see this observe that if x is any real number, then $10^x \in \mathbb{R}^+$. Moreover, $\log_{10} 10^x = x$, then if y is any element of $\mathbb{R}$, $10^y\theta = \log 10^y = y$ and hence $\mathbb{R}^+\theta = \mathbb{R}$.

(ii) $(x+y)\theta = e^{x+y} = e^x.e^{\,y} = (x\theta)(y\theta)$ and so θ is a homomorphism. If $x\theta = y\theta$ then $e^x = e^y$ and $e^{x-y} = 1$ from which $x - y = 0$ and $x = y$, so θ is one to one. Is θ onto? yes, for if y is any positive real number the equation $e^x = y$ has a solution $x \in \mathbb{R}$. Thus θ is an isomorphism between $(\mathbb{R},+)$ and $(\mathbb{R}^+,\cdot)$.