The Open University of Sri Lanka
B.Sc. Degree Programme – Level 05
Final Examination - 2010/2011
Pure Mathematics
PMU 3291/PME 5291 – Complex Analysis

Duration: - Three Hours

Date: - 27-12-2010. **Time: - 9.30 a.m. – 12.30 p.m.**

Answer FIVE questions only.

$\mathbb{C}$ denotes the set of all complex numbers, and $\mathbb{R}$ denotes the set of all real numbers.

1.

(a) Let z and w are complex numbers.

Prove that:

(i) if $|w|=1$ and $\bar{z}w \neq 1$, then $\left|\dfrac{z-w}{1-\bar{z}w}\right|=1$.

(ii) if $|z|<1$ and $|w|<1$, then $\left|\dfrac{z-w}{1-\bar{z}w}\right|<1$.

(b) Sketch the following sets and determine which are domains:

(i) $|2-i|=|z-1|$,

(ii) $|z|\leq 1$,

(iii) $|z|<|z-4|$.

2.

(a) Show that $|(3+i)-2-4i|=\sqrt{10}\,|z-(1+i)|$ represents a circle and find its centre and radius.

(b) Prove using the $\varepsilon-\delta$ definition that $\lim\limits_{z\to(1+i)}(3+i)z=2+4i$.

(c) Let $f(z)=|z-i|^2$ for all $z\in\mathbb{C}$. For any $z_0\in\mathbb{C}$, show that

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{\Delta z\to 0}\left(\left(\overline{z}_0+i\right)+\frac{\overline{\Delta z}}{\Delta z}(z_0-i)\right).$$

Hence, prove that f is differentiable only at $z=i$.

3.

(a) Show that a necessary condition that a function $f(z)=u(x,y)+v(x,y)$ is differentiable in a domain D is that $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=\dfrac{\partial v}{\partial x}$ through out D.

(b) Show that the function $f(z)=\left(3x^2+y\right)+i\left(2y^3-x\right)$ is not analytic at any point.

(c) Verify that the function $u(x,y)=y^3-3x^2y$ is harmonic in the entire complex plane.

Find a harmonic conjugate $v(x,y)$ for $u(x,y)=y^3-3x^2y$.

Is $u(x,y)$ a harmonic conjugate for $v(x,y)$? Justify your answer.

4.

(a) Evaluate $\int_{\gamma} Im(z-i)dz$,

where γ is the contour defined by the circular arc γ_1 along $|z|=2$ from $z=2$ to $z=2i$ followed by the line segment γ_2 from $z=2i$ to $z=0$.

(b) Prove that

$$\oint_C \frac{dz}{(z-z_0)^{n+1}}=\begin{cases}0 & if\ n\neq 0\\ 2\pi i & if\ n=0\end{cases},$$

where n is an integer and C is a circle of radius r and center at z_0.

(c) Find an upper bound for $\left|\oint_C \frac{z+e^z}{z^2+1}dz\right|$,

where C is the positively oriented circle $|z|=2$.

5.

(a) Expand $f(z)=\dfrac{3z+1}{z(1-z)}$ in a Laurent series valid for:

(i) $0<|z|<1$,

(ii) $1<|z-2|$.

(b) Show that $z=-1$ is a removable singularity of the function $f(z)=\dfrac{z^3}{z(z+1)}$.

(c) Give an example of a function $f(z)$ analytic in the plane except at $z=0$ and $z=1$, which has removable singularity at $z=0$ and an essential singularity at $z=1$.

6.

(a) Suppose you are given the real integral

$$I=\int_0^{2\pi}\frac{d\theta}{5/4+\sin\theta}.$$

Show that the substitution $z=e^{i\theta}$ convert it into

$$I=2\oint_{|z|=1}\frac{dz}{(z+i/2)(z+2i)}.$$

Appling residue theorem, show that $I=8\pi/3$.

(b) Evaluate

$$\int_0^{2\pi}\frac{x\sin x}{(x^2+1)(x^2-4)}dx.$$

7.

(a) State the Maximum modulus Theorem.

Deduce that if a non – constant analytic function $f(z)$ has no zero in a domain D, then $|f(z)|$ does not attain its minimum value in D.

(b) Let $S=\{z\in\mathbb{C} \mid |z|\leq 2\}$ and $f(z)=2z+5i$ for all $z\in S$.

(i) Show that $|f(z)|=\sqrt{41+20Im\,z}$, if $|z|=2$.

(ii) Find the maximum and minimum of $|f(z)|$ on S.

8.

(a) Prove that if f is an analytic function in a domain D containing z_0, and $f'(z_0)\neq 0$ then $w=f(z)$ is a conformal mapping at z_0.

(b) Determine where each of the following complex mappings is conformal:

(i) $f(z)=z^3-3z+1$,

(ii) $f(z)=z-e^{-z}+1-i$.

(c) Show that the transformation $w=e^{i\lambda}\left(\dfrac{z-\alpha}{z-\bar{\alpha}}\right)$, maps the upper half plane $Im\,z\geq 0$ onto the unit disk $|w|\leq 1$, where $\lambda\in\mathbb{R}$ and $Im\alpha>0$.

Show further that if $0<\lambda<2\pi$ and the points $z=0$ and $z=1$ are to be mapped onto the points $w=1$ and $w=e^{i\lambda/2}$, respectively, then the transformation is given by

$$w=e^{i\lambda/2}\left(\frac{z+e^{-i\lambda/2}}{z+e^{i\lambda/2}}\right).$$

****