The Open University of Sri Lanka
B.Sc. Degree Programme – Level 05
Open Book Test (OBT) - 2010/2011
Pure Mathematics / Computer Science
PMU 3294/PME 5294/CSU 3276 – Discrete Mathematics

Duration: - One & Half Hours

Date: - 24-09-2010. **Time: - 4.00 p.m. – 5.30 p.m.**

Answer All Questions.

01. (a) Write down the negation of each of the following statements in clear and concise English. Do not use the expression "It is not the case that" in your answers.

(i) "Three distinct points lying in a plane form a triangle or three distinct points lying in a plane are collinear".

(ii) "If a and b are odd natural numbers then ab is an odd natural number and $a+b$ is an even natural number".

(b) Let p and q be two statements, by means of a truth table show that $p \wedge (\sim(\sim p \vee q)) \vee (p \wedge q)$ is logically equivalent to p.

(c) Determine the truth value of each of the following statements. Justify your answer.

(i) $\lim_{\theta \to \infty} \frac{\sin\theta}{\theta} = 1$ if and only if $\sin^2\theta + \cos^2\theta > 1$.

(ii) For each $x \in \mathbb{R}$, $5^{2x} - 5^{x+1} + 6 < 0$.

(iii) For each $x \in \mathbb{R}$, for each $y \in \mathbb{R}$, $\frac{x+y}{2} \geq \sqrt{xy}$.

02. Prove or disprove each of the following statements. Name the method of your proof.

(i) For each $n \in \mathbb{N}$, $n^2 + n + 41$ is a prime number.

(ii) If the product of two integers a and b is an odd natural number then a is an odd natural number and b is an odd natural number.

(iii) There exists $n \in \mathbb{N}$ such that $6n - 1$ and $6n + 1$ are not prime numbers.

(iv) For each $n \in \mathbb{N}$, $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$.

03. (a) Let A and B are two non-empty subsets of a universal set U, prove that

$$(A \times B) \cap (A \times C) = A \times (B \cap C).$$

(b) Let R be the relation defined on $\mathbb{Z}$ by $_m R_n$ if m divides n for all $m, n \in \mathbb{Z}$.

Show that R is not a partial order.

(c) Define R on $\mathbb{R}$ by $_x R_y$ if $x - y \in \mathbb{Q}$ for every $x, y \in \mathbb{R}$. Determine whether R is an equivalence relation on $\mathbb{R}$.

04. (a) Use a Cayley composition table to show that the set of functions $G = \left\{ x, -x, \frac{1}{x}, -\frac{1}{x} \right\}$ under the binary operation of composition of functions forms an abelian group.

(b) Let G be the group of real numbers under the usual addition, and let G' be the group of positive real numbers under the usual multiplication. Show that the mapping $f : G \to G'$, defined by $f(a) = 2^a$, is a homomorphism.

Is it an isomorphism? Justify your answer.

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Model Answer

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01. (a) (i) "Three distinct points lying in a plane do not form a triangle and three distinct points lying in a plane are not collinear."

(ii) "a and b are odd natural numbers and ab is an even natural number or $a+b$ is an odd natural number."

(b) Let p and q be two statements. Then consider the following truth table:

p	q	$\sim p$	$\sim p \vee q$	$\sim(\sim p \vee q)$	$p \wedge q$	$\sim(\sim p \vee q) \vee (p \wedge q)$	$p \wedge (\sim(\sim p \vee q)) \vee (p \wedge q)$
T	T	F	T	F	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F

The truth values of the column headed by $p \wedge (\sim(\sim p \vee q)) \vee (p \wedge q)$ are the same as the truth values of the column headed by p.

Thus $p \wedge (\sim(\sim p \vee q)) \vee (p \wedge q)$ is logically equivalent to p.

(c) (i) $\lim_{\theta \to \infty} \frac{\sin\theta}{\theta} = 1$ iff $\sin^2\theta + \cos^2\theta > 1$.

$\lim_{\theta \to \infty} \frac{\sin\theta}{\theta} = 1$ is false $\left(\because \lim_{\theta \to \infty} \frac{\sin\theta}{\theta} = \infty\right)$

$\sin^2\theta + \cos^2\theta > 1$ is false $\left(\because \sin^2\theta + \cos^2\theta = 1\right)$

Therefore the biconditional statement is true.

(ii) $\forall x \in \mathbb{R}, \quad 5^{2x} - 5^{x+1} + 6 < 0$.

When $x = 0$; $\quad 1 - 5 + 6 = 2 > 0$

Therefore the statement is false.

(iii) $\forall x \in \mathbb{R}, \quad \forall y \in \mathbb{R}, \quad \frac{x+y}{2} \geq \sqrt{xy}$

When $x = -1$ and $y = -1$;

$$\left.\begin{array}{l} \frac{x+y}{2} = \frac{(-1)+(-1)}{2} = -1 \\ \sqrt{xy} = \sqrt{(-1)(-1)} = 1 \end{array}\right\} \quad \frac{x+y}{2} \not\geq \sqrt{xy} \qquad \forall x, \ \forall y$$

Therefore the statement is false.

02. (i) $\forall n \in \mathbb{N}, \ n^2 + n + 41$ is prime.

When $n = 41$; $41^2 + 41 + 41 = 41(41+1+1) = 41 \times 43$ is not prime.

Therefore we have disproved the statement using counter example.

(ii) If ab is odd, then a is odd and b odd.

Proof by Contraposition.

If a is even or b is even, then ab is even.

Case (i) If a is even and b is not even (odd).

Then $a = 2k$ and $b = 2k' + 1$ for some $k, k' \in \mathbb{Z}$

$ab = (2k)(2k'+1) = 2\left[k(2k'+1)\right]$ is even.

Case (ii) If a is not even (odd) and b is even,

Then $a = 2k + 1$ and $b = 2k'$ for some $k, k' \in \mathbb{Z}$

$ab = (2k+1)(2k') = 2\left[(2k+1)k'\right]$ is even.

Case (iii) If a is even and b is even.

Then $a = 2k$ and $b = 2k'$ for some $k, k' \in \mathbb{Z}$

$ab = (2k)(2k') = 2\left[k(2k')\right]$ is even.

(iii) $\exists n \in \mathbb{N}$ such that $6n-1$ and $6n+1$ are not prime.

When $n=4$, $6n-1=23$ and $6n+1=25$ (not prime)

That is $(6n-1)$ and $(6n+1)$ are not prime.

Proof by illustrative example.

(iv) $\forall n \in \mathbb{N}$ $1.1!+2.2!+3.3!+......+n.n!=(n+1)!-1$

Proof by Mathematical Induction. Now consider $\sum_{k=1}^{n} k.k! = (n+1)!-1$.

When $n=1$; $1.1!=(1+1)!-1$. Thus the result is true for $n=1$.

Assume that the result is true for $n=p$;

That is $\sum_{k=1}^{p} k.k! = (p+1)!-1$.

Now consider when $n=p+1$;

$$\begin{aligned}\sum_{k=1}^{p+1} k.k! &= \sum_{k=1}^{p} k.k! + (p+1).(p+1)! \\ &= (p+1)!-1+(p+1)(p+1)! \\ &= (p+1)![1+p+1]-1 \\ &= (p+2)!-1\end{aligned}$$

03. (a) $$\begin{aligned}(A\times B)\cap(A\times C) &= \{(x,y) \mid (x,y)\in A\times B \wedge (x,y)\in A\times C\} \\ &= \{(x,y) \mid (x\in A \wedge y\in B)\wedge(x\in A\wedge y\in C)\} \\ &= \{(x,y) \mid x\in A\wedge(y\in B\cap C)\} \\ &= A\times(B\cap C)\end{aligned}$$

(b) $R=\{(m,n) \mid m,n\in\mathbb{Z}\wedge m|n\}$ is not anti-symmetric.

$2,-2\in\mathbb{Z}$ and $2|-2 \wedge -2|2$ but $2\neq -2$

That is ${}_2R_{-2} \wedge {}_{-2}R_2$ but $2\neq -2$

Therefore R is not a partial order.

(c) $R = \{(x, y) | x, y \in \mathbb{R} \wedge x - y \in \mathbb{Q}\}$

R is reflexive:

$$\forall x \in \mathbb{R}, \quad x - x = 0 \in \mathbb{Q}$$

R is symmetric:

$$ {}_xR_y \Rightarrow x - y \in \mathbb{Q}$$
$$\Rightarrow -(x-y) = y - x \in \mathbb{Q}$$
$$\Rightarrow {}_yR_x$$

R is transitive:

$$ {}_xR_y \wedge {}_yR_z \Rightarrow x - y \in \mathbb{Q} \wedge y - z \in \mathbb{Q}$$
$$\Rightarrow (x-y) + (y-z) = (x-z) \in \mathbb{Q}$$

$\therefore$ R is an equivalence relation on $\mathbb{R}$.

04. (a) $G = \left\{x, -x, \frac{1}{x}, -\frac{1}{x}\right\}$ Binary operation $\circ$ is composition of functions.

$\circ$	x	$-x$	$\frac{1}{x}$	$-\frac{1}{x}$
x	x	$-x$	$\frac{1}{x}$	$-\frac{1}{x}$
$-x$	$-x$	x	$-\frac{1}{x}$	$\frac{1}{x}$
$\frac{1}{x}$	$\frac{1}{x}$	$-\frac{1}{x}$	x	$-x$
$-\frac{1}{x}$	$-\frac{1}{x}$	$\frac{1}{x}$	$-x$	x

From the above Cayley composition table,

(i) The operation $\circ$ satisfies the closure property.

(ii) Since composition of functions is associative, $\circ$ is associative.
(We could also verify this directly from the table)

(iii) x is the identity element.

(iv) Pairs of inverse elements are;

$\{x, x\}; \{-x, -x\}, \left\{\frac{1}{x}, \frac{1}{x}\right\}, \left\{-\frac{1}{x}, -\frac{1}{x}\right\}$

(v) Since the above table is symmetric about the main diagonal,

$\circ$ satisfies the commutative property.

$\therefore$ (G, $\circ$) is an abelian group.

(b) $f : (\mathbb{R}, +) \to (\mathbb{R}^+, \times)$ such that $f(a) = 2^a$

Take $a, b \in (\mathbb{R}, +)$, then

$f(a+b) = 2^{a+b} = 2^a \times 2^b$

Therefore f is a homomorphism.

Let $f(a) = f(b)$

$2^a = 2^b$

$a = b$

$\therefore f$ is 1−1

Now take any $x \in (\mathbb{R}^+, \times)$ such that $f(a) = x$ $(\in \mathbb{R}^+)$

Then $2^a = x$

$\Rightarrow \ln_2 x = a$ $(\in \mathbb{R})$

$\therefore f$ is onto.

Since f is bijective, and homomorphism.

$\therefore f$ is an isomorphism.