The Open University of Sri Lanka
B.Sc. Degree Programme – Level 3
Department of Physics
Waves in Physics - PYU 1162 / PYE 3162
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MODEL ANSWERS

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1. (a)

$$
\begin{aligned}
\left(4-\sqrt{5}\,\mathrm{j}\right)^3 &= \left(4-\sqrt{5}\,\mathrm{j}\right)^2 \times \left(4-\sqrt{5}\,\mathrm{j}\right)\\
&= \left(16-8\sqrt{5}\,\mathrm{j}+5\,\mathrm{j}^2\right) \times \left(4-\sqrt{5}\,\mathrm{j}\right)\\
&= \left(11-8\sqrt{5}\,\mathrm{j}\right) \times \left(4-\sqrt{5}\,\mathrm{j}\right) \qquad [\text{ because } \mathrm{j}^2=-1\ ]\\
&= 44-32\sqrt{5}\,\mathrm{j}-11\sqrt{5}\,\mathrm{j}+40\,\mathrm{j}^2\\
&= 44-32\sqrt{5}\,\mathrm{j}-11\sqrt{5}\,\mathrm{j}-40\\
&= 4-43\sqrt{5}\,\mathrm{j}
\end{aligned}
$$

Magnitude of $\left(4-\sqrt{5}\,\mathrm{j}\right)^3$ is $= \sqrt{(4)^2+(43\sqrt{5})^2}$
$= \sqrt{16+9245}$
$= \underline{96.234}$

Direction of $\left(4-\sqrt{5}\,\mathrm{j}\right)^3$ is $= \arctan\left(\frac{-43\sqrt{5}}{4}\right)$
$= \sqrt{16+9245}$
$= \underline{\underline{87.62^0}}$

(b)

$$
\begin{aligned}
\frac{Ae^{j(\omega t+\frac{\pi}{2})}}{4+5j} &= \frac{A(\cos\left(\omega t+\frac{\pi}{2}\right)+j\sin\left(\omega t+\frac{\pi}{2}\right)}{4+5j}\\
&= \frac{A\left[\cos\left(\omega t+\frac{\pi}{2}\right)+j\sin\left(\omega t+\frac{\pi}{2}\right)\right]}{4+5j} \times \frac{4-5j}{4-5j}\\
&= \frac{A\left[4\cos\left(\omega t+\frac{\pi}{2}\right)+4j\sin\left(\omega t+\frac{\pi}{2}\right)-5j\cos\left(\omega t+\frac{\pi}{2}\right)+5\sin\left(\omega t+\frac{\pi}{2}\right)\right]}{4^2+5^2}\\
&= \frac{A\left[4\cos\left(\omega t+\frac{\pi}{2}\right)+5\sin\left(\omega t+\frac{\pi}{2}\right)\right]+j\,A\left[4\sin\left(\omega t+\frac{\pi}{2}\right)-5\cos\left(\omega t+\frac{\pi}{2}\right)\right]}{41}
\end{aligned}
$$

Real part is $= \frac{A}{41}\left[4\cos\left(\omega t+\frac{\pi}{2}\right)+5\sin\left(\omega t+\frac{\pi}{2}\right)\right]$

Imaginary part is $= j\,\frac{A}{41}\left[4\sin\left(\omega t+\frac{\pi}{2}\right)-5\cos\left(\omega t+\frac{\pi}{2}\right)\right]$

(c) We know that, $e^{j\theta}=\cos\theta+j\sin\theta$

(i) $Z_1 = [j]^j = \left[e^{j\left(\frac{\pi}{2}\right)}\right]^j = \left[e^{j\left(\frac{\pi}{2}\pm 2n\pi\right)}\right]^j$

$= e^{j^2\left(\frac{\pi}{2}\pm 2n\pi\right)} = e^{-\left(\frac{\pi}{2}\pm 2n\pi\right)}$

$\approx$ 0.208, 3.88×10^{-4}, 1.11×10^{2}, …. (when n = 0. 1, 2, …)

(ii) $Z_2 = [j]^{8.03} = \left[e^{j\left(\frac{\pi}{2}\pm 2n\pi\right)}\right]^{8.03}$

$= e^{j\left[8.03 \times \left(\frac{\pi}{2}\pm 2n\pi\right)\right]}$

$= \cos\left[8.03 \times \left(\frac{\pi}{2} \pm 2n\pi\right)\right] + j\ \sin\left[8.03 \times \left(\frac{\pi}{2} \pm 2n\pi\right)\right]$

$\approx$ 0.999 + 0.047 j, 0.972 + 0.233 j, …. (when n = 0. 1, …)

2. (a) (i) Periodic Motion or Harmonic Motion: Any motion that repeats itself in regular interval.

Displacement, x, at a time is given by $x(t) = x_m \cos(\omega t + \varphi)$
Or $x(t) = x_m \sin(\omega t + \varphi)$

Amplitude (x_m): Magnitude of maximum displacement of the particle in either direction.

Angular frequency (ω)**:** Number of complete cycles per second.
$\omega = \frac{2\pi}{T} = 2\pi f$ (Unit: rad s^{-1}) where f is frequency.

Phase (φ)**:** Phase is a constant and it represents a stage in a cycle that cyclic motion has reached at a given time.

(ii) **Velocity – SHM**

$v(t) = \frac{dx}{dt} = -\omega x_m \sin(\omega t + \varphi)$

Velocity amplitude, $v_m = \omega x_m$

$v(t) = \frac{dx}{dt} = -v_m \sin(\omega t + \varphi)$

Acceleration – SHM

$a(t) = \frac{dv}{dt} = -\omega^2 x_m \cos(\omega t + \varphi)$

Acceleration amplitude, $a_m = \omega^2 x_m$

$a(t) = \frac{dx}{dt} = -a_m \cos(\omega t + \varphi)$

Also, $a(t) = -\omega^2 x(t)$

(iii) Hooke's law: Force = $-kx$ and Newton's second law: Force = ma = $-m\omega^2 x$

Force = $-m\omega^2 x = -kx$

where k is spring constant, *and* $k = m\omega^2$

Therefore, the angular frequency (ω) is

$$\omega^2 = \frac{k}{m} \qquad \text{and } \omega = \sqrt{\frac{k}{m}}$$

Therefore, the period (T) is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

(b) (i) Displacement: $x(0) = x_m \cos\varphi = -8.50$ cm (1)

Velocity: $v(0) = -\omega x_m \sin(\varphi) = -0.92 \text{ m s}^{-1}$ (2)

Acceleration: $a(0) = -\omega^2 x_m \cos(\varphi) = +47.0 \text{ m s}^{-2}$ (3)

(3)/(1) ==> $-\omega^2 = 47.0 \ / \ -0.085$

$$\omega = \sqrt{\frac{47.0}{0.085}} = \underline{\underline{23.52 \text{ rad s}^{-1}}}$$

$$f = \frac{\omega}{2\pi} = \frac{23.52 \times 7}{2 \times 22} = \underline{\underline{3.74 \text{ Hz}}}$$

(ii) (2)/(1) ==> $-\omega \tan(\varphi) = -0.92 / -0.085$

$\tan(\varphi) = -0.92 / (0.085 \times 23.52) = -0.46$

$\varphi = \tan^{-1}(-0.46) = -24.71^{\circ}$ or 155.29°

But, $\varphi = \underline{\underline{155.29^{\circ}}}$ *(See next part for explanation)*

(iii) From (1), $x_m = x(0) / \cos\varphi = -0.085 / \cos(-24.71)$

$= -0.094$ m

But amplitude is a positive constant. Therefore, φ cannot be -24.71°

Therefore, $x_m = x(0) / \cos\varphi = -0.085 / \cos(155.29)$

$= \underline{\underline{0.094 \text{ m}}}$

3. (a) Let the oscillations of the spring be along the x axis. Then the spring force and damping force acting on the mass are:

$$F_{restoring} = -kx$$

$$F_{damping} = -bv = -b\frac{dx}{dt}$$

The net force is, $F_{net} = m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$

Hence the differential equation of motion for free oscillations of the system is,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Or $$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Or $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0 \qquad \textit{where } \omega_o = \sqrt{\frac{k}{m}} \textit{ and } \gamma = \frac{b}{m}$$

(b) Damped frequency is 0.995 times the ω_o.

Equation for the damping frequency is $\omega^2 = \omega_o^2 - \frac{\gamma^2}{4}$

$$0.995^2\omega_o^2 = \omega_o^2 - \frac{\gamma^2}{4}$$

$$0.99\,\omega_o^2 = \omega_o^2 - \frac{\gamma^2}{4}$$

$$0.01\,\omega_o^2 = \frac{\gamma^2}{4} = \frac{b^2}{4m^2}$$

Therefore, $b^2 = 0.04\,m^2\omega_o^2 = 0.04\,m^2\frac{k}{m} = 0.04\,mk$

$$b = 0.2\sqrt{mk} = 0.2\sqrt{0.2 \times 80} = \underline{\underline{0.8 \text{ kg s}^{-1}}}$$

(c) $$Q = \frac{\omega_o}{\gamma} = \sqrt{\frac{k}{m}} \times \frac{m}{b} = \sqrt{\frac{km}{b^2}}$$

$$Q = \sqrt{\frac{80 \times 0.2}{0.8^2}} = 5$$

Time taken for 4 complete cycle = $4 \times \frac{2\pi}{\omega_o}$

For damped oscillations, amplitude at any time *t* is = $A(t) = A_o e^{-\frac{\gamma t}{2}}$

Therefore, the factor by which amplitude is reduced after 4 complete cycle is

$$\frac{A(t)}{A_o} = e^{-\frac{\gamma t}{2}} = e^{-\frac{b8\pi}{2m\omega_o}} = e^{-\frac{0.8 \times 4\pi}{0.2 \times 20}}$$

$$= \underline{\underline{0.081}}$$

(d) Decay of energy of a damped oscillation system is

$$E(t) = E_o e^{-\gamma t}$$

Therefore, the fraction of the original energy is left after 4 oscillations is,

$$\frac{E(t)}{E_o} = e^{-\gamma t}$$

$$= \underline{\underline{0.0066}}$$

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