THE OPEN UNIVERSITY OF SRI LANKA
B.Sc. DEGREE PROGRAMME - Level 4 - 2009/2010
PHU 2142 / PHE 4142: Advanced Electromagnetism

Assignment – 1 **MODEL ANSWERS**

1. (a) Computing $\nabla \cdot (\nabla \wedge \boldsymbol{A})$ in cartesian coordinates,

$$\nabla \wedge \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{e}_z,$$

$$\nabla \cdot \nabla \wedge \boldsymbol{A} = \frac{\partial^2 A_z}{\partial x\,\partial y} - \frac{\partial^2 A_y}{\partial x\,\partial z} + \frac{\partial^2 A_x}{\partial y\,\partial z} - \frac{\partial^2 A_z}{\partial x\,\partial y} + \frac{\partial^2 A_y}{\partial x\,\partial z} - \frac{\partial^2 A_x}{\partial y\,\partial z} = 0.$$

(b) Computing $\nabla \cdot (\nabla \wedge \boldsymbol{A})$ in cartesian coordinates,

$$\nabla(\psi) = \frac{\partial \psi}{\partial x}\boldsymbol{e}_x + \frac{\partial \psi}{\partial y}\boldsymbol{e}_y + \frac{\partial \psi}{\partial z}\boldsymbol{e}_z$$

$$\nabla \wedge (\nabla(\psi)) = \left(\frac{\partial^2 \psi}{\partial y\,\partial z} - \frac{\partial^2 \psi}{\partial z\,\partial y}\right)\boldsymbol{e}_x + \left(\frac{\partial^2 \psi}{\partial z\,\partial x} - \frac{\partial^2 \psi}{\partial x\,\partial z}\right)\boldsymbol{e}_y + \left(\frac{\partial^2 \psi}{\partial x\,\partial y} - \frac{\partial^2 \psi}{\partial y\,\partial x}\right)\boldsymbol{e}_z = \boldsymbol{0}.$$

This vanishes identically because $\frac{\partial^2 \psi}{\partial y\,\partial z} - \frac{\partial^2 \psi}{\partial z\,\partial y} = 0$ for a "well-behaved" function.

(c) In cartesian coordinates,

$$\boldsymbol{A} \cdot \nabla = A_x\,\partial_x + A_y\,\partial_y + A_z\,\partial_z,$$

so

$$(\boldsymbol{A} \cdot \nabla)\,\boldsymbol{B} = (A_x\,\partial_x + A_y\,\partial_y + A_z\,\partial_z)(B_x\,\boldsymbol{e}_x + B_y\,\boldsymbol{e}_y + B_z\,\boldsymbol{e}_z) =$$
$$(A_x\,\partial_x B_x + A_y\,\partial_y B_x + A_z\,\partial_z B_x)\,\boldsymbol{e}_x + (A_x\,\partial_x B_y + A_y\,\partial_y B_y + A_z\,\partial_z B_y)\,\boldsymbol{e}_y + (A_x\,\partial_x B_z + A_y\,\partial_y B_z + A_z\,\partial_z B_z)\,\boldsymbol{e}_z$$

(d) Consider $(A_x\,\partial_x B_x + A_y\,\partial_y B_x + A_z\,\partial_z B_x)$.

$$(A_x\,\partial_x B_x + A_y\,\partial_y B_x + A_z\,\partial_z B_x) = \frac{1}{\sqrt{x^2+y^2+z^2}}\left(x\,\partial_x \frac{x}{\sqrt{x^2+y^2+z^2}} + y\,\partial_y \frac{x}{\sqrt{x^2+y^2+z^2}} + z\,\partial_z \frac{x}{\sqrt{x^2+y^2+z^2}}\right) =$$

$$-\frac{x y^2}{(x^2+y^2+z^2)^{3/2}} + x\left(\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}}\right) - \frac{x z^2}{(x^2+y^2+z^2)^{3/2}} = 0.$$

By symmetry, this must hold for the other two components so $(\hat{\boldsymbol{r}} \cdot \nabla)\,\hat{\boldsymbol{r}} = \boldsymbol{0}$.

2. $\underline{\boldsymbol{E}} = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}$

(a) Expressing it in cartesian coordinates, we get,

$$\underline{\boldsymbol{E}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\,\boldsymbol{(x,\ y,\ z)}$$

Therefore, $\nabla \times \underline{\boldsymbol{E}}$ = curl of $\boldsymbol{E}$ = $\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right) \times \frac{q}{4\pi\varepsilon_0}\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\,\boldsymbol{(x, y, z)}$

Curl of cartesian coordinates in this form can be shown as (0, 0, 0).

(Students are expected to work this out)

Therefore, $\nabla \times \underline{\boldsymbol{E}} = 0$

(b) Because the electric field is spherically symmetric, we can write

$$E \bullet dl = E_r \hat{r} \bullet dl = E_r dr$$

And hence the integral depends only on the radii, r_a and r_b. Therefore,

$$\int_a^b E \bullet dl = \int_{r_a}^{r_b} E_r dr$$

$$= \frac{q}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{1}{r^2}dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

(c) Consider the above result of an integration:

$$\int_a^b E \bullet dl = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$$

For a closed path, $r_a = r_b$. Hence if you substitute this in the above result, it becomes zero.

And hence integral around any closed path is zero.

(d) Stokes theorem is, for any surface S:

$$\oint E \bullet dl = 0 = \int_S \vec{\nabla}\times\vec{E} \bullet dS$$

Hence, $\nabla \times \underline{E} = 0$

(e) From (a) or (d), we know that, for a point charge, $\nabla \times \underline{E} = 0$

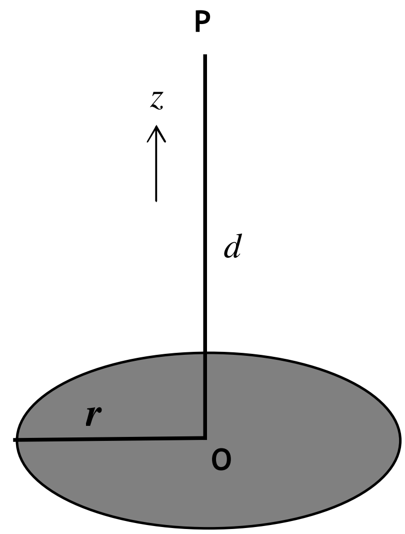
Static charges may be considered as a distribution of number of point charges.

Principle of superposition states that the total filed due to many charges is vector sum of their individual fields, and hence can be given by,

$$\boldsymbol{E = E_1 + E_2 + E_3 + \ldots\ldots..}$$

Therefore, it can be concluded that, for any static charge distribution $\nabla \times \underline{E} = 0$

3. We know that the electric field at a distance d on an axis (say z- axis) of a ring carrying a charge q is



$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left(d^2+r^2\right)^{3/2}}\hat{z}$$

Therefore, in this case of a disk, consider a small ring element of inner radius s and outer radius $s + ds$.

Hence the charge on this element is

$$= \sigma\,(2\pi s\, ds).$$

$$= \frac{Q}{\pi r^2}\,(2\pi s\, ds).$$

Then the electric field dE at P due to this element is,

$$dE = \frac{Q}{\pi r^2}\,(2\pi s\, ds) \times \frac{1}{4\pi\varepsilon_0}\frac{d}{\left(d^2+s^2\right)^{3/2}}\hat{z}$$

$$= \frac{Qd}{2\pi\varepsilon_0 r^2}\frac{s.ds}{\left(d^2+s^2\right)^{3/2}}\hat{z}$$

(a) Therefore, the electric field $\underline{\boldsymbol{E}}$ at the point P, at a distance d above the centre of the disk of radius r is,

$$E(d) = \int_0^r dE = \frac{Qd}{2\pi\varepsilon_0 r^2}\int_0^r \frac{s.ds}{\left(d^2+s^2\right)^{3/2}}\hat{z}$$

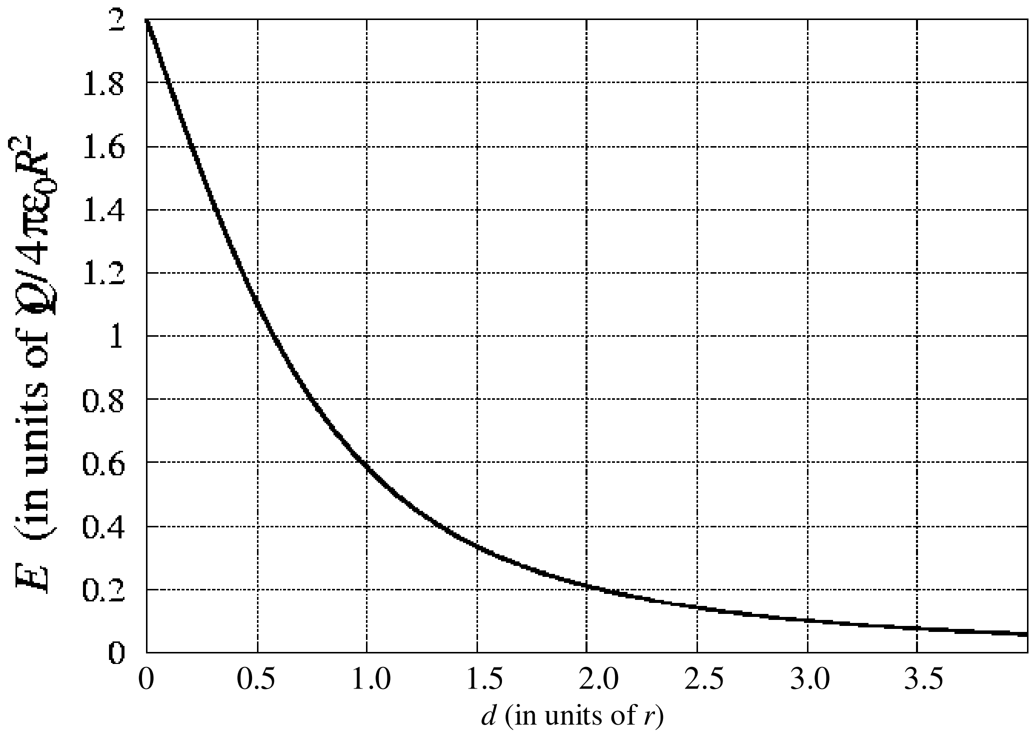
$$= \frac{Qd}{2\pi\varepsilon_0 r^2}\left[\frac{-1}{\left(d^2+s^2\right)^{1/2}}\right]_0^r\hat{z}$$

$$= \frac{Qd}{2\pi\varepsilon_0 r^2}\left[\frac{1}{d} - \frac{1}{\left(d^2+r^2\right)^{1/2}}\right]\hat{z}$$

(b) For all positive d*'s,* the above equation can be re-written as,

$$\frac{E(d)}{(Q/4\pi\varepsilon_0 r^2)} = 2\left[1 - \frac{d/r}{\left(1+(d/r)^2\right)^{1/2}}\right]\hat{z}$$

Then the plot for varying d would look like,



(c) $$E(d) = \frac{Q}{2\pi\varepsilon_0 r^2}\left[1 - \frac{d/r}{\left(1+(d/r)^2\right)^{1/2}}\right]\hat{z}$$

When $d << r$, we have,

$$E(d) = \frac{Q}{2\pi\varepsilon_0 r^2}\hat{z}$$ (because other terms become negligibly small)

4. (a) Refer OUSL book 1A.

(b) Refer OUSL book 1A.

(c) From (b)(ii) $B = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+b^2)^{3/2}}$ Weber / m^2

where, a – radius of the coil; b- distance along the axis.

At the center means, $B = \frac{\mu_0 I}{2a}$ Weber / m^2

For N turns, $B = \frac{\mu_0 NI}{2a} = \frac{4\pi \times 10^{-7} \times 10 \times 5}{2 \times 5 \times 10^{-2}}$ = 6.28 x 10^{-4} Weber / m^2

5. (a) The magnetic field inside a wire, that is carrying a current I, at a distance r from the axis is,

$$B = \frac{\mu_0 I r}{2\pi . R^2}$$

Then the magnetic field energy density u is,

$$u = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{\mu_0 I^2 r^2}{8\pi^2 . R^4}$$

Now, volume element of a cylindrical shell element of radius r and thickness dr is,

$$= 2\pi r\, l\, dr$$

Therefore the magnetic field energy inside a length l of the wire of radius R is,

$$= \int_0^R \frac{\mu_0 I^2 r^2}{8\pi^2 . R^4} \times 2\pi r\, l\, dr$$

$$= \frac{\mu_0 I^2 l}{4\pi . R^4} \int_0^R r^3 dr$$

$$= \frac{\mu_0 I^2 l}{16\pi}$$

(b) The magnetic field energy inside a self-inductor (of self-inductance L) is,

$$= \frac{1}{2} L I^2 = \frac{\mu_0 I^2 l}{16\pi}$$

Therefore, $L = \dfrac{\mu_0 l}{8\pi}$

The contribution of the interior portion of the conductor per unit length to the total self-inductance is, $\dfrac{\mu_0}{8\pi}$

(c) The time dependent current in an LR circuit connected to a battery of voltage V_o at time $t = 0$ is,

$$I(t) = \frac{V_o}{R}\left(1 - e^{-t/\tau}\right) \qquad \text{where } \tau = L/R = (0.09)/(0.05) = 1.8 \text{ s}$$

Using the known equations and substituting the values,

(i) 5.4 s

(ii) 2,300 J

(iii) 10,600 J

6. (a) Consider $\nabla \times \underline{B} = \mu_o \varepsilon_o \dfrac{\partial \underline{E}}{\partial t}$

Take curl on both side: $\nabla \times \nabla \times \underline{B} = \mu_o \varepsilon_o \nabla \times \dfrac{\partial \underline{E}}{\partial t}$

But, $\nabla \times \nabla \times \underline{B} = \nabla (\nabla \cdot \underline{B}) - \nabla^2 \underline{B}$

Then, $\nabla (\nabla \cdot \underline{B}) - \nabla^2 \underline{B} = \mu_o \varepsilon_o \dfrac{\partial (\nabla \times E)}{\partial t}$

By substituting equations from given sets, the wave equation for ***B*** can be given as,

$$\nabla^2 \underline{B} = \mu_o \varepsilon_o \frac{\partial^2 \underline{B}}{\partial t^2}$$

(b) Taking divergence on both sides of $\nabla \times \underline{E} = - \dfrac{\partial \underline{B}}{\partial t}$, we get

$$\nabla \cdot \nabla \times \underline{E} = - \nabla \cdot \frac{\partial \underline{B}}{\partial t} = \frac{\partial (\nabla . B)}{\partial t} = 0$$

Similarly,

$$\nabla \cdot \nabla \times \underline{B} = \nabla \cdot \mu_o \varepsilon_o \frac{\partial \underline{E}}{\partial t} = \mu_o \varepsilon_o \frac{\partial (\nabla . E)}{\partial t} = 0$$

Therefore, divergences of the second set equations (both) are consistence.

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