The Open University of Sri Lanka
B.Sc. Degree Programme – Level 05
Final Examination- 2010/2011
Pure Mathematics / Computer Science
PMU 3294/ CSU 3276/ PME 5294 – Discrete Mathematics

Duration: - Three Hours

Date: - 04-01-2011. **Time: - 1.30 p.m. – 4.30 p.m.**

Answer FIVE Questions only.

1. (a) Let p and q be two statements. By using a truth table, show that the conditional statement $p \Rightarrow q$ is logically equivalent to the disjunctive statement $\sim p \vee q$.
Deduce that the negation of $(p \Rightarrow q)$ is $p \wedge \sim q$.

(b) Without using truth table, show that the inverse and converse of a conditional statement are contra positive of each other.

(c) Rewrite the following statements without using the conditional:
(i) If it is cold then he wears a hat,
(ii) Productivity increases therefore wages rise,
(iii) He is poor only if he is happy.

(d) Write the negation of each of the following statements:
(i) $-2 < x \leq 5$,
(ii) For each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x \in \mathbb{R}$, for each $y \in \mathbb{R}$, if $|x-y| < \delta$ then $|f(x)-f(y)| < \varepsilon$,
(iii) $x^2 > 1$ and $x > 0$ if and only if $x > 1$,
(iv) If $\sin x \neq \frac{1}{\sqrt{2}}$ and $\sin x \neq -\frac{1}{\sqrt{2}}$ then $\sin^2 x > \frac{1}{2}$ or $\cos^2 x > \frac{1}{2}$,
(v) For each Archemedian ordered field there is an elementarily equivalent ordered field that is not Archimedian ordered.

2. Prove or disprove each of the following statements and name the method of your proof in each case:
   (a) Every continuous function is differentiable,
   (b) For each $n \in \mathbb{N}$, $17^n - 10^n$ is divisible by 7,
   (c) $\sum_{n=1}^{\infty} r^n$ is divergent implies that $|r| \geq 1$,
   (d) For each $x \in \mathbb{R}$, for each $y \in \mathbb{R}$, $\frac{x+y}{2} \geq \sqrt{xy}$,
   (e) There exists $x \in \mathbb{R}$ such that $x^{i\pi} + 1 = 0$,
   (f) Let a, b be real numbers. If $a+b \geq 6$ then $a \geq 3$ or $b \geq 3$.

3. (a) Let R be the binary relation on $A = \{2, 3, 4, 6, 9\}$ defined by "x is relatively prime to y", i.e. the only positive divisor of x and y is 1.
   (i) Write R as a set of ordered pairs and represent it in a directed graph,
   (ii) Find the domain and range of R,
   (iii) Is R a transitive? Justify your answer.

   (b) Let A be a set on which an equivalence relation R is defined. Prove that any two equivalence classes of A are either identical or disjoint.

   (c) Let m be an integer. Define R on $\mathbb{Z}$ by ${}_xR_y$ if m divides $x-y$. Show that R is an equivalence relation on $\mathbb{Z}$.
   If $m = 2$, show that $C_0 = \{2n: \; n \in \mathbb{Z}\}$ and $C_1 = \{2n+1: \; n \in \mathbb{Z}\}$ are the only two equivalence classes of R.

4. (a) Let G be a group of non-zero complex numbers under usual multiplication and G' be a group of non-zero real numbers under usual multiplication. Show that the mapping $f: G \to G'$, defined by $f(z) = |z|$, is a homomorphism.
   Hence describe geometrically the *kernel* of the homomorphism f.
   If the binary operation in the above two groups is changed to usual addition, determine whether g is a homomorphism.

(b) Let the binary operation $*$ for combining sets is defined as

$A * B = (A \cup B) - (A \cap B)$. Show that G, consisting of the set of all subsets of a set S, together with the binary operation $*$, forms a group.

(**Note:** You may assume that the associative property is satisfied).

If S is the set of natural numbers, solve the equation $\{1, 2, 4\} * X = \{3, 4\}$.

5. (a) Prove that the number of ways in which n distinct objects can be distributed into k boxes, $B_1, B_2, \ldots, B_k$ such that there are r_i objects in box B_i for $i = 1, 2, \ldots, k$, is $\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$.

(b) Let there be five flavors of ice cream: banana, chocolate, lemon, strawberry and vanilla. How many variations will there be, if you can have three scoops

(i) with different flavors,

(ii) and you are not worry about flavors?

(c) (i) Find the number of ways that ten toys can be divided among three children if the youngest child is to receive four toys and each of the others three toys.

(ii) Let a box B contains seven marbles numbered 1 through 7. Find the number of ways of drawing from B first two marbles, then three marbles, and lastly the remaining two marbles.

(ii) In how many ways can the fifteen students take three different tests if five students are to take each of the tests?

6. (a) If A and B are independent events in a probability space S, show that the following pairs of events are also independent.

   (i) $\{A^c, B\}$ (ii) $\{A^c, B^c\}$, where A^c is the complement event of A.

   (b) An anti-aircraft gun can fire four shots at a time. The probabilities of the first, second, third and the last shot hitting the enemy aircraft are $0\cdot7$, $0\cdot6$, $0\cdot5$ and $0\cdot4$ respectively. Find the probability that four shots aimed at on enemy aircraft will bring the aircraft down.

   (c) Teams A and B play in a cricket series and the team that first wins four games wins the series. The team that wins the first game also wins the third game, and the team that wins the second game also wins the fourth game.

   (i) Construct the tree diagram such that the series can occur.

   (ii) Let the probability of the team A wins a game be $0\cdot6$. What is the probability that the team B wins the series?

7. (a) Draw a graph having the given properties or explain why such a graph does not exist.

   (i) Simple graph of three edges and four vertices having degrees 2, 3, 1, 2.

   (ii) Simple graph of six vertices having degrees 2, 2, 3, 4, 4, 6.

   (iv) Simple graph of three edges and four vertices but all the vertices have degree at least 2.

   (b) Let G be a graph with set of four vertices $\{v_1, v_2, v_3, v_4\}$, whose adjacency matrix

   A is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

   (i) Without drawing the diagram of G, determine whether G is connected.

   (ii) Find the number of paths of length three joining v_2 and v_4 and name all those paths.

   (iii) Write down all the components of G.

   (iv) Is G a forest? Justify your answer.

8. (a) Consider the Fibonacci sequence problem "How may pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on".

Let f_n be the number of pairs of rabbits in the n^{th} month.

(i) Find a difference equation satisfied by f_n, and initial conditions that define the sequence $\{f_n\}$,

(ii) Compute the total number of pairs of rabbits in the 8^{th} month,

(iii) Solve the difference equation in (i) and hence compute $\lim_{n\to\infty} f_n$.

(iv) Show that $r_{n+1} = \dfrac{1}{1+r_n}$, where $r_n = \dfrac{f_n}{f_{n+1}}$,

(v) By using (iv) and assuming that $\lim_{n\to\infty} r_n$ exists, show that $\lim_{n\to\infty} r_n = \dfrac{\sqrt{5}-1}{2}$.

(b) Find the general solution of each of the following difference equations:

(i) $f_{n+2} + 2f_{n+1} + 4f_n = 0$,

(ii) $f_{n+2} - 2f_{n+1} + f_n = n^2 + 1$.