The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination- 2014/2015
Pure Mathematics - Level 04
PUU2143 /PUE4143– Differentiable Functions

Duration: - Two hours

Date: - 09-11-2015 **Time: - 1:30pm. – 3:30pm.**

Answer **Four** questions only.

1. State the $\varepsilon - \delta$ definition for differentiability of a real valued function at a given point.

Use this definition to prove each of the following.

(i) Let $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \in \mathbb{Q}^c \end{cases}$. Then f is not differentiable at 0.

(ii) Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$. Then f is differentiable at 0.

(iii) Let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$. Then f is not differentiable at 0.

2. Prove or disprove each of the following statements.

(i) If real valued function f is differentiable at a point c, then f is continuous at c.

(ii) If real valued function f is continuous at a point c, then f is differentiable at c.

(iii) If $f(x) = \begin{cases} 2x & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$, then f is differentiable at 0.

(iv) There is a real function defined on $\mathbb{R}$, which is differentiable only at 0.

3. (a) State, without proof, the Rolle's Theorem.

Suppose $c_0, c_1, \dots, c_{n-1}, c_n$ are real numbers such that $c_0 + \frac{c_1}{2} + \cdots + \frac{c_{n-1}}{n} + \frac{c_n}{n+1} = 0$.

Prove that the equation $c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} + c_n x^n = 0$ has at least one real root between 0 and 1.

(b) State and prove the Mean Value Theorem for derivatives.

Prove that there exists $c \in (-1,1)$ such that $6c^5 + 5c^4 + 2c = 1$.

4. Let $f(x)$ be the function given by

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+f(x)}} \quad \text{for } x \geq 0.$$

Show that

(i) $\frac{1}{4} \leq f(x) \leq \frac{1}{2}$.

(ii) $\lim_{x\to 0^+} f(x) = \frac{1}{4}$ and $\lim_{x\to\infty} f(x) = \frac{1}{2}$.

5. (a) State, without proof, the intermediate value property for derivatives.

Let $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$. Prove that f is not a derivative of any real function.

(b) Expand $x^8 + x^5$ in powers of $(x-1)$. {Hint: Find the appropriate Taylor polynomial}

6. Evaluate each of the following limits:

(i) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$,

(ii) $\lim_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$,

(iii) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$,

(iv) $\lim_{x\to\infty}\left(x - \sqrt{(x-1)(x-2)}\right)$.