The Open University of Sri Lanka
B.Sc. /B.Ed. Degree Programme
Final Examination - 2014/2015
Pure Mathematics - Level 04
PUU2144/PUE4144 – Group Theory I

Duration: Two Hours

Date: 25.10.2015 **Time: 9.30 a.m. – 11.30 a.m.**

Answer Four Questions Only.

(01). (a). Let $G = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$ be a set.

(i). Show that G is a group under matrix addition.

(ii). Is the group G commutative? Justify your answer.

(b). (i).Find all subgroup of $\mathbb{Z}_8$ and draw the lattice diagram.

(ii).Determine order of each element of group in 0, 1, 2, 3, 4, 5 under addition modulo 6.

(02). (a). (i). The permutation σ is given by $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 6 & 2 & 1 & 7 & 3 \end{pmatrix}$, write σ as a product of disjoint cycles.

(ii). What is the order of σ?

(iii). Is σ an even permutation ? Justify your answer.

(b). State Lagrange's Theorem.

G is a group of order 12 and H is a group of order 3. What is the index of H in G.

(c). Find the order of the element (3, 3) in $\mathbb{Z}_6 \times \mathbb{Z}_8$.

(03). (a). Define normal subgroup of a group G.

Let $H \leq G$, then show that following statements are equivalent.

(i). $H \trianglelefteq G$

(ii). $gHg^{-1} = H \quad \forall g \in G$

(iii). $gHg^{-1} \subseteq H \quad \forall g \in G$

(iv). $ghg^{-1} \in H \quad \forall g \in G \; and \; \forall h \in H$

(b). If $H \leq G$ such that $|G:H| = 2$ then prove that $H \trianglelefteq G$.

(c). G is a group of order 11. Prove that G is a simple group.

(04). (a). If H is a subgroup of G and K is a normal subgroup of G, then prove each of the following.

(i). $H \cap K$ is a normal subgroup of H.

(ii). If H is a normal subgroup of G, then HK is a normal subgroup of H.

(b). Let G be a group. The normalizer of a set S of a group G is define by

$$N_G(S) = \{g \in G \mid gSg^{-1} = S\}$$

Show that $N_G(S)$ is a subgroup od G.

(05). (a). Let $(G, *)$ and (G_0, o) be a two groups. If $\phi: G \to G_0$ be a homomorphism and let 1 and 1_o be the identity element of G and G_0 respectively. Prove each of the following.

(i). $\phi(1) = 1o$,

(ii). $\phi(g^{-1}) = (\phi(g))^{-1} \quad \forall g \in G$,

(iii). $\ker \phi \trianglelefteq G$.

(b). Let $f: G \to H$ and $g: H \to K$ be two group homomorphisms. Show that the function composition $gof: G \to K$ is a homomorphism from G to K.

(c). Let $G$and H be two groups. Show that the mapping $f: G \to H$ defined by $f(x) = e_H$ is a homomorphism.

(06). (a). State the definition of *kernel* of a group homomorphism.

Let $\theta: G \to G_0$ be a group homomorphism.

Let K be the *kernel* of a homomorphism θ. Define the map $\phi: {}^{G}/_{K} \to \theta(G)$ by $gK \mapsto \theta(g)$.

(i). Show that ϕ is well-defined.

(ii). Show that ϕ is bijective.

(iii). Prove that ϕ is a homomorphism.

(b). State and prove *Second Isomorphism Theorem.*