The Open University of Sri Lanka
Department of Mathematics and Computer Science
B.Sc/ B.Ed Degree Programme
Final Examination - 2015/2016
Applied Mathematics– Level 05
APU3244/ APE5244– Graph Theory

DURATION: - THREE HOURS

Date: - 19 – 01 – 2017 **Time: - 1.30 p.m. – 4.30 p.m.**

ANSWER FIVE QUESTIONS ONLY

01. (a) Disprove each of the following statements by drawing a suitable graph:

(i) Every regular graph is a complete graph,

(ii) All connected graphs have *Hamiltonian* paths,

(iii) Complete bipartite graphs are complete graphs,

(iv) If the line graph of a graph is *Hamiltonian* then the underlying graph is *Hamiltonian*.

(b) Prove each of the following statements:

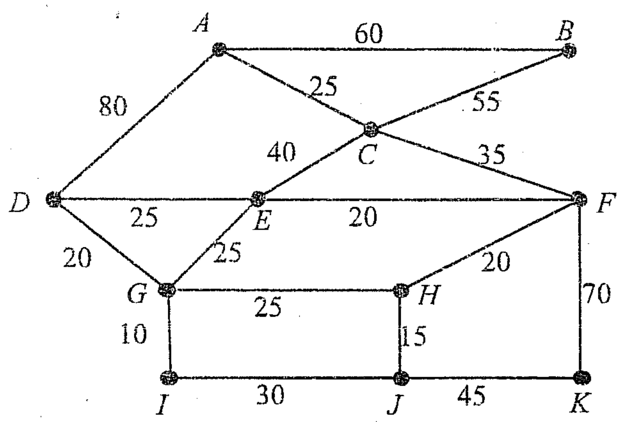
(i) A complete graph with n vertices contain $\frac{n(n-1)}{2}$ edges,

(ii) If a connected graph is *Eulerian* then the degree of each vertex of the graph is even,

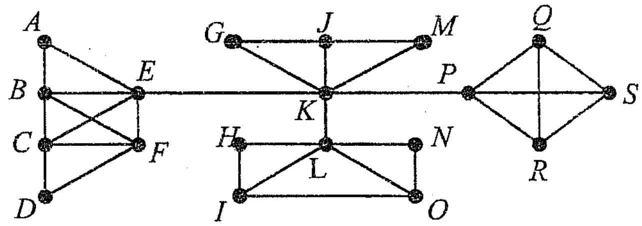
(iii) If each cycle of a connected graph has even length then the graph is bipartite.

02. (a) Find a spanning tree of the complete graph K_5 by applying the *breadth first search algorithm*. Hence, deduce the general type of the spanning tree of the complete graph K_n.

(b) The edges in the following graph represent roads between cities A, B, C, D, E, F, G, H, I, J, K and these roads are not paved. The values on the edges denote the lengths of roads. The authorities decide to pave these roads so that there is a path of paved roads between each pair of cities. Which roads should be paved so that the paved road length is minimum?

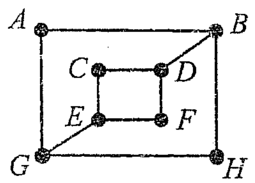


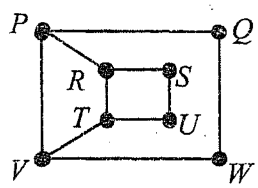
(c) Use the *depth first search algorithm* to produce a spanning tree of maximum height, by choosing '*A*' as the root.



Hence, find the height of the spanning tree.

03. (a) Determine whether the following two graphs are *isomorphic* or not. Justify your answer.

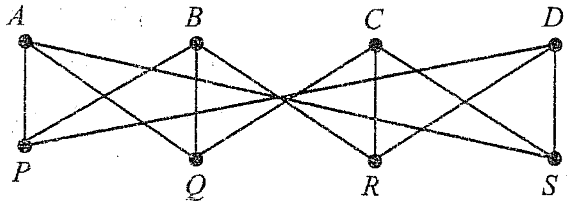




(b) Find a route from the city *A* that visits each of the cities *B*, *C*, *D* and *E* exactly once and return to the starting point with the least total bus fare. The following table gives the least bus fare available for a journey between two cities.

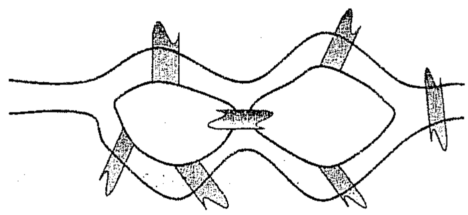
City	*A*	*B*	*C*	*D*	*E*
A	-	110	100	160	50
B	110	-	90	140	150
C	100	90	-	60	130
D	160	140	60	-	120
E	50	150	130	120	-

(c) Show that the following graph is *planar* by drawing it in the plane without crossing edges.



Verify the *Euler's formula* for the planar drawing of the above graph.

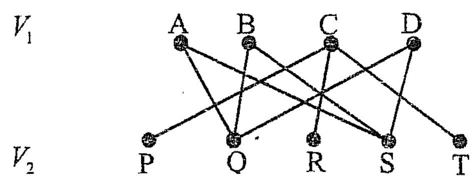
(d) Can someone cross each bridge shown in the following map exactly once and returned to the starting point? Justify your answer.



04. Let $G(V_1, V_2)$ be a bipartite graph. Define a *complete matching* from V_1 to V_2.

(a) State the *Hall's theorem*.

(b) Explain why the following graph has no *complete matching* from V_1 to V_2.



When does the *Hall's theorem* fail in the above graph?

(c) Suppose that a new company has 5 employees: Kelum, Dinusha, Loshini, Chalani and Vidura. Each employee will assume only one of 6 responsibilities: planning, publicity, sales, marketing, development and analyzing. Each employee is capable of doing one or more of these jobs: Kelum could do planning, sales, marketing, or analyzing; Dinusha could do planning or development; Loshini could do publicity, sales or analyzing; Chalani could do planning, sales or analyzing; and Vidura could do planning, publicity, sales or analyzing.

(i) Model the capabilities of these employees by using a bipartite graph,

(ii) Find three different assignments so that each employee is assigned one responsibility,

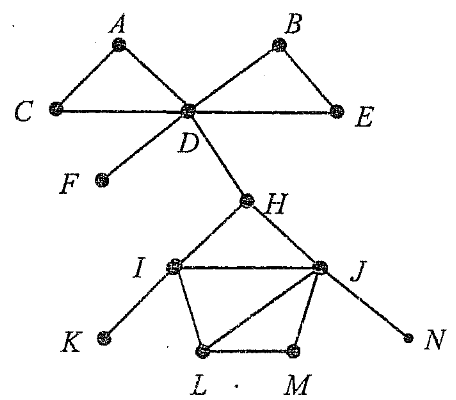
(iii) Can all the jobs be filled by capable employees? Justify your answer.

05. Define a *block* of a connected graph and state two properties of a *block*.

(a) Find the maximum number of *cut points* in a connected graph with $p\ (>1)$ points.

(b) Verify, by giving an example, that if a vertex of a graph is a *cut point* then it is not a *cut point* of its complement graph.

(c) Consider the following graph G.



(i) Find all the *cut points* in G and draw the *cut point graph*,

(ii) Are there any *bridges* in G? Justify your answer,

(iii) Find all *blocks* of G and draw the *block graph*.

06. Let E be a non- empty finite set and let $S_1, S_2, \ldots, S_m$ be m non-empty subsets of E.

Define a *transversal* of a family $\Im = (S_1, \ldots, S_m)$.

(a) State the *Konig- Egervacy theorem*

(b) Let $E = \{1,2,3,4,5,6\}$ and let $S_1 = \{1,2\}, S_2 = \{1,2\}, S_3 = \{2,3\}, S_4 = \{2,3\}$ and $S_5 = \{1,4,5,6\}$.

(i) Determine whether $\Im = (S_1, \ldots, S_5)$ has transversal or not. Justify your answer,

(ii) List three *partial transversals* of $\Im' = (S_1, S_2, S_4, S_5)$,

(iii) Write down the *incidence matrix* A of the family $\Im$.

Hence, find the *term rank*,

(iv) Verify the *Kong- Egervacy theorem* for the *incidence matrix* A.

Hence, verify your result obtained in part (i).

07. Let B be a family of m-point subsets of a set X with n points, such that any l points lie in at most one member of B.

Define a *Steiner Triple System* of order n (STS(n)).

(a) Show that $|B| \le \frac{{}^{n}C_{l}}{{}^{m}C_{l}}$.

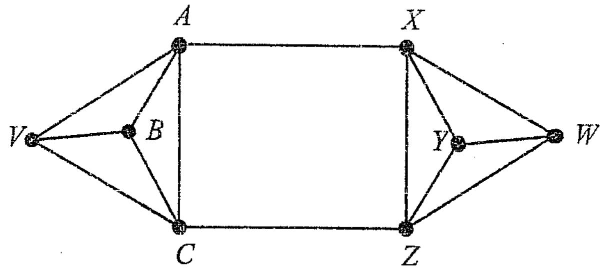
Under what condition, the equality holds?

(b) Nine school girls walk each day in 3 groups of 3. Arrange girls walk for 4 days so that in that time, each pair of girls walks together in a group just once.

08. Let G be a connected graph and let V and W be any two distinct vertices of G.

State the *Menger's theorems.*

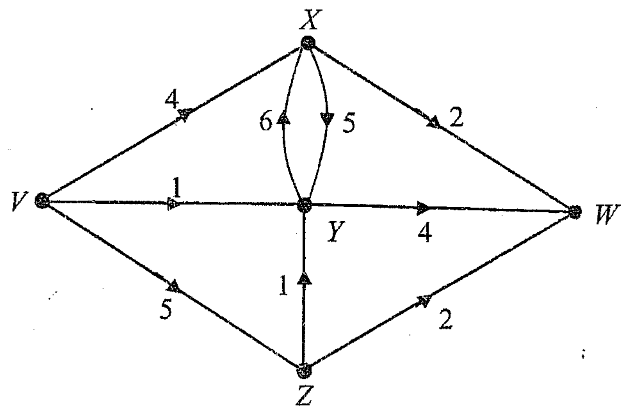
(a) Let G be the following graph.



(i) Write down all the *edge-disjoint paths* and *vertex- disjoint paths* from V to W in the graph G,

(ii) Find a *minimal* VW *- disconnecting* set and VW *- separating sets* in the graph G. Hence, verify the *Menger's theorems.*

(b) Let N be the following network.



Verify the *maximum flow - minimum cut theorem* for the network N.