The Open University of Sri Lanka
B.Sc./B.Ed Degree Programme /Continuing Education Programme
Final Examination 2016/2017
Applied Mathematics – Level 04
APU 2143/APE4143 – Vector Calculus
Duration :- Two Hours.

Date :-27.07.2017 | **Time:- 1.00 p.m. - 3.00 p.m.**

Answer Four Questions Only.

1. (a) State and sketch the domain of the function $f(x, y) = \ln(y - x)$.

   (b) Sketch the level curves of the function $f(x, y) = \ln(y - x)$.

   (c) Let $f_1(x, y) = \dfrac{y}{\sqrt{x^2 + y^2}}$ and $f_2(x, y) = \dfrac{x}{1 + \sqrt{x^2 + y^2}}$. Find the following limits, if they exist:

   (i) $\lim\limits_{(x,y)\to(0,0)} f_1(x, y)$, (ii) $\lim\limits_{(x,y)\to(0,0)} f_2(x, y)$.

   (d) Discuss the continuity at the origin of the functions $f_1(x, y)$ and $f_2(x, y)$.

2. (a) If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$ then show that

   (i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$,

   (ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \tan^3 u$.

   (b) Define a stationary point of a single valued function $f(x, y)$ defined over a domain D. Explain briefly how you could determine the nature of the stationary point.

   (c) Find the maximum and minimum values of the function $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$ and determine their nature.

3. (a) Prove that $\text{grad}\,\phi$ is a vector normal to the contour surface $\phi(x, y, z) = c$, where c is a constant.

   (b) (i) Show that the equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point

$P(x_0, y_0, z_0)$ is given by $(x-x_0)\left(\frac{\partial F}{\partial x}\right)_P + (y-y_0)\left(\frac{\partial F}{\partial y}\right)_P + (z-z_0)\left(\frac{\partial F}{\partial z}\right)_P = 0$.

(ii) Using the above result, find the equation of the tangent plane to the surface $F(x, y, z) = 2xyz + x^2 y$ at the point $(2, 3, 4)$.

(c) Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z=47$ at $(4, -3, 2)$.

4. (a) State Gauss' Divergence theorem.

(b) Verify the above theorem considering the vector field $\underline{F}=(x^3-yz)\underline{i}-2x^2y\underline{j}+z\underline{k}$ taken over the entire surface of the cube given by $0 \le x \le a$, $0 \le y \le a$ and $0 \le z \le a$.

(c) Let $\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$ and $r = |\underline{r}|$ then prove that

(i) $\underline{\nabla}\cdot\underline{r} = 3$, (ii) $\underline{\nabla}\cdot r\underline{r} = 4r$, (iii) curl $(\underline{a}\times\underline{r}) = 2\underline{a}$.

5. (a) State Stokes' Theorem.

(b) Verify Stokes' Theorem considering the vector field $\underline{F}=(x+y)\underline{i}+(y-x)\underline{j}+z^3\underline{k}$ defined over the hemispherical region bounded by $x^2+y^2+z^2=1$ and $z \ge 0$.

(c) Prove that the vector field $\underline{F}=(x^2-yz)\underline{i}+(y^2-zx)\underline{j}+(z^2-xy)\underline{k}$ is irrotational and find a corresponding scalar potential function ϕ such that $\underline{F}=\underline{\nabla}\phi$.

6. (a) Suppose that S is a plane surface lying in the xy-plane, bounded by a closed curve C. If $\underline{F} = P(x, y)\underline{i} + Q(x, y)\underline{j}$ then show that $\oint_C (Pdx+Qdy) = \iint_S \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$.

(b) Verify the above result for the integral $\oint_C (3x^2-8y^2)dx+(4y-6xy)dy$, where C is the closed curve of the region in the first quadrant bounded by $y=0$, $x+y=1$ and $x=0$.