The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination - 2019/2020
Pure Mathematics - Level 05
PEU5303 –Number Theory

Date: 05.02.2021 **Time 2.00 p.m. - 4.00p.m.**

Answer four questions only

Question (01)

(a)

(i) State the Well Ordering Principle.

(ii) Let $A = \{a+b+c+d:\ a,b,c,d \in \mathbb{N},\ a^2+b^2+c^2+d^2 = 1000\}$ and $B = \{z+x+y:\ x,y,z$ are positive odd integers such that $x^2+3y^2 = z^2\}$

Prove that the set A has a least element but the set B does not have a least element.

(b)

(i) Use Euclidean Algorithm to find $x,\ y \in \mathbb{Z}$ such that $34x + 21y = 1$.

(ii) People of a certain country use only two kinds of coins, one worth \$34 and the other \$21. A person of that country buys a toy that worth \$4 from a toy shop. By assuming buyer and seller has enough coins of both kinds (\$34 and \$21), suggest a way to exchange the coins to pay the amount of the toy. (*Hint*: use part (i))

(c)

A school marching band has more than 100 members, but fewer than 300 members. When they line up in rows of 4 there is one extra student, when they line up in rows of 5 there are two extra students and when they line up in rows of 7 there are three extra students.

Use Chinese Remainder Theorem to find all possible values for the number of students in the band.

Question (02)

(a) Prove that if p is a prime greater than 3, then $p^2 - 1$ is divisible by 24.

(b) Let p be a prime number. Prove that $p + 4$ and $p + 8$ are also prime only when $p = 3$.

(c) Prove that an integer of the form $4k + 3$, where $k \in \mathbb{Z}$, cannot be written as a sum of two square numbers.

(d) Prove that if m and n are odd integers not divisible by 3, then $24|(m^2 - n^2)$.

Question (03)

(a) Show that $11^{84} - 5^{84}$ is divisible by 7.

(b) Find all values of $n \in \mathbb{N}$ such that $3^n + 2^n$ divisible by 7. Prove your answer.

(c) Determine the remainder when 3^{372} is divided by 37.

(d)

(i) Prove that there are infinitely many integers k such that $k^2 + k + 1$ is divisible by 7.

(ii) Also, show that there is no integer k such that $k^2 + k + 1$ is divisible by 5.

Question (04)

(a) Prove the following.

(i) For any $k \in \mathbb{N}$, $\sqrt{4k + 2}$ is irrational. (*Hint*: Follow the steps of the proof of $\sqrt{2}$ is irrational.)

(ii) Any integer of the form $4k + 2$ is never a perfect cube (i.e., third power of an integer).

(b)

(i) Prove the Euclid's Lemma. I.e., prove that if $a|bc$ then $a|c$, for any a, b, c integers with $\gcd(a, b) = 1$.

(ii) Prove that for any prime p, $\sqrt{p}$ is irrational. (*Hint*: Follow the steps of the proof of $\sqrt{2}$ is irrational and use the Euclid's Lemma.)

Question (05)

(a) Let a, b, c, x and y be integers. Also, let n be a positive integer. Prove the following.

(i) If $a|b$ and $a|c$ then $a|bx + cy$.

(ii) $\gcd(a, b) = \gcd(a - b,\ b)$

(iii) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $ac \equiv bd \pmod{n}$.

(b) Let $\overline{abcd}$ be the digital representation of a positive integer M, where $1000 \leq M \leq 9999$. I.e., $M = \overline{abcd} = 10^3a + 10^2b + 10c + d$, where $a, b, c, d \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with $a \neq 0$.

(i) Prove that $11|M$ if and only if $11|a - b + c - d$.

(ii) Prove that $7|M$ if and only if $7|\ a - 2b - 3c - d$.

(iii) Find $k, l, m, n \in \{1, 2, 3, 4, 5, 6, -1, -2, -3, -4, -5, -6\}$ such that $13|M$ if and only if $13|ka + lb + mc + nd$.

Question (06)

(a) State, without proof, the Fermat's little theorem.

(b) Prove that if a is a positive integer then a always has the same last digit as its fifth power a^5.

(c) Prove that if a an integer then $255|a(a^{16} - 1)$.

(d) Prove that if a is an odd integer then $32|a(a^{16} - 1)$.

(e) Prove that $10 \nmid [(a - 1)! + 1]$, for any positive integer a.