00269

The Open University of Sri Lanka

B.Sc/B.Ed. DEGREE, CONTINUING EDUCATION PROGRAMME

Final Examination 2019/2020

Level 03 Pure Mathematics

PEU3202/PUU1142 Vector Spaces

Duration: - Two Hours

Date: - 18-02-2021 **Time: 1.30 p.m. to 3.30 p.m.**

Answer four questions only

1.

(a) Suppose V is a vector space over a field F. Prove that if $\alpha(x+y)=\alpha(x+z)$ for some $\alpha \in F-\{0\}$ then $y=z$ for all $x, y, z \in V$.

(b) Show that the set $\mathbb{R}^2$ is not a vector space over the field $\mathbb{R}$ under the operations given in (i) and (ii). For any $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$

(i) $(a_1, b_1) + (a_2, b_2) = (0, a_2 + b_2)$; $\alpha(a_1, b_1) = (a_1, b_1)$

(ii) $(a_1, b_1) + (a_2, b_2) = (a_1 + b_1, a_2 + b_2)$; $\alpha(a_1, b_1) = (\alpha^2 a_1, \alpha^2 b_1)$

(c) Determine whether the set $V = \{(a, b) \mid a, b \in \mathbb{N}\}$ is a subspace of $\mathbb{R}^2$ over the field $\mathbb{R}$ under usual addition and scalar multiplication.

2.

(a) Suppose W_1 and W_2 are subspaces of a vector space V over a field F.

(i) Prove that $W_1 \cap W_2$ is a subspace of the vector space V over the field F.

(ii) Define $W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$. Prove that $W_1 + W_2$ is a subspace of the vector space V over the field F.

(b) Prove that if $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$ are linearly dependent then $a_1 b_2 = a_2 b_1$, where, $\mathbb{R}^2$ is the vector space over $\mathbb{R}$, under usual addition and scalar multiplication.

3.

(a) Suppose V is a vector space over a field F and S is a non-empty subset of V. Prove that $L_s = Sp < S >$ is the smallest subspace of V containing S.

(b) Suppose W is a subspace of a finite dimensional vector space V over the field F, then prove that dim W = dim V if and only if $W = V$

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Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^2$. Note that V and W are vector spaces over the field $\mathbb{R}$ under the usual addition and scalar multiplication.

Consider the mapping $T : V \to W$ defined by $T(x, y, z) = (x - y - z, x + y + z)$.

(i) Show that T is a linear transformation.
(ii) Find the kernel of T.
(iii) Is T an Isomorphism? Justify your answer.
(iv) Find the image of T and prove that image of T is a subspace of the vector space W.

5.

(a) Suppose that V and W are vector spaces over a field F and $T : V \to W$ is a linear transformation. Prove that $T(-x) = -T(x)$ for all $x \in V$.

(b) Let $V = \mathbb{R}^3$. Note that V is a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication.

Let $T : V \to V$ be linear transformation defined by $T(a, b, c) = (c, b, a)$.

Determine whether the following sets are invariant subspaces of the vector space V over the field $\mathbb{R}$ under T:

(i) $U = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$
(ii) $W = \{(a, a, a) \mid a \in \mathbb{R}\}$

(c) Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$ where $u, v \in \mathbb{R}^3$.
Define $< u, v > = {x_1}^2 - {x_2}^2 - x_1 x_3$. Is $< u, v >$ an inner product on $\mathbb{R}^3$? Justify your answer.

6.

(a) Define Euclidian Space.

(b) State and prove the Schwarz inequality.

(c) Show that the three vectors $u_1 = (1, 2, 2)$, $u_2 = (1, -1, 2)$ and $u_3 = (1, 0, 1)$ form a basis for E^3, the usual Euclidean three space. Construct an orthonormal basis for E^3 out of $\{u_1, u_2, u_3\}$ using the Gram–Schmidt process.