00177

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	**: Mathematics**
Level	**: 04**
Name of the Examination	**: Final Examination**
Course Title and - Code	**: Group Theory 1 – PEU4303**
Academic Year	**: 2020/21**
Date	**: 22/03/2022**
Time	**: 9.30 a.m. To 11.30 a.m.**
Duration	**: Two Hours.**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **(6)** questions in **(2)** pages.
3. Answer any 04 questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary
6. Involvement in any activity that is considered as an exam offense will lead to punishment
7. Use blue or black ink to answer the questions.
8. Clearly state your index number in your answer script

01. Let X be the set $\mathbb{R}/\{0, \quad 1\}$. Let f_i , $i = 1,2,3 \ldots .6$ be mappings of $X \to X$ defined by

$f_1(x) = x$, $f_2(x) = \frac{1}{x}$, $f_3(x) = 1 - x$, $f_4(x) = \frac{1}{1-x}$, $f_5(x) = \frac{x}{1-x}$, $f_6(x) = 1 - \frac{1}{x}$.

Show that

(a) $f_2 \circ f_3 = f_4$ (b) $f_4 \circ f_6 = f_1$.

Construct a Cayley table for the set $M = \{f_1, \quad f_2, \quad f_3, \quad f_4, \quad f_5, \quad f_6\}$

with respect to the operation compositions of mappings, entering $f_i \circ f_j$ in the row opposite f_i and column below f_j .

Hence, show that $(M, \circ)$ is a group. Show also that $(M, \circ)$ is non-abelian.

02. (a) Which of the following subsets of S_4 are subgroups? Justify your answers.

(i) $\{I, \quad (1 \quad 2 \quad 3), \quad (1 \quad 3 \quad 2)\}$

(ii) $\{I, \quad (1 \quad 2 \quad 3), \quad (1 \quad 3 \quad 2), \quad (1 \quad 3 \quad 4), \quad (1 \quad 4 \quad 3)\}$

(iii) $\{I, \quad (1 \quad 2 \quad 3 \quad 4), \quad (1 \quad 3)(2 \quad 4), \quad (1 \quad 4 \quad 3 \quad 2)\}$.

(b) Let H , K be subgroups of a group G. Prove that $H \cap K$ is a subgroup of G.

03.(a) Let $(G, \quad \cdot)$ be a group. Prove each of the followings:

(i) The identity element of G is unique.

(ii) Every element $a \in G$ has a unique inverse element $a^{-1} \in G$.

(b) Let $(G, \quad *)$ be a group with identity e and $a, b \in G$. Verify the following identities.

(i) For any $m \in \mathbb{Z}$, $a^{-1}(ab)^m a = (ba)^m$.

(ii) For any $m \in \mathbb{Z}$, $(aba^{-1})^m = ab^m a^{-1}$.

04. (a) Find the index of each of the following subgroups in $(\mathbb{Z}, +)$.

(i) $2\mathbb{Z} = \{2r, r \in \mathbb{Z}\}$

(ii) $5\mathbb{Z} = \{5r, r \in \mathbb{Z}\}$

(b) Determine the elements of the cyclic group generated by the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ explicitly.

(c) Let a, b be elements of a group G. Assume that a has order 5 and that $a^3 b = ba^3$.

Prove that $ab = ba$.

05. (a) Prove that the subgroup N of G is a normal subgroup of G if and only if

$gN = Ng, \forall\, g \in G$.

(b) Determine which of the following subgroups are normal in the given group.

(i) $H = \{0, 3\}$ in $(\mathbb{Z}_6, +)$

(ii) $K = \{I, \ (1\ \ 2)\}$ in S_3.

06. (a) If ϕ is a homomorphism of G *into* $\bar{G}$ with kernel $Ker\phi = K$.

Prove that $K \lhd G$.

(b) Prove that a homomorphism ϕ of G *into* $\bar{G}$ is one to one if and only if $Ker\phi = e$.

(c) A map ϕ: $\langle \mathbb{Z}, + \rangle \rightarrow \langle \mathbb{C}^*, \cdot \rangle$ is determined by $\phi(n) = i^n$.

Show that ϕ is a homomorphism and find $Ker\phi$.