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057

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc. / B. Ed Degree Programme

Department	**: Mathematics**
Level	**: 04**
Name of the Examination	**: Final Examination**
Course Title and - Code	**: Differentiable Functions – PEU4316**
Academic Year	**: 2020/21**
Date	**: 01.04.2022**
Time	**: 2.00 p.m. To 4.00 p.m.**
Duration	**: Two Hours**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **(6)** questions in **(2)** pages.
3. Answer any **(4)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary.
6. Involvement in any activity that is considered as an exam offense will lead to punishment.
7. Use blue or black ink to answer the questions.
8. Clearly state your index number in your answer script.

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1. (a). State the $\varepsilon - \delta$ definition of a differentiable function.

(i). Let f be a real-valued function defined by $f(x) = 2x^2 + 1$ for $x \in \mathbb{R}$.
Show that f is differentiable at $x = 2$ and $f'(2) = 8$.

(ii). Let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right); x \neq 0, \\ 0 \qquad\quad ; x = 0. \end{cases}$$

Show that f is differentiable at $x = 0$ and $f'(0) = 0$.

(b). Let f be a real-valued function and c be a real number. Show that if f is differentiable at c, then f is continuous at c.
State whether the converse of the above statement is true.
Justify your answer.

2. (a). Let $f(x)$ and $g(x)$ be functions and c be a real number such that for some $\delta > 0$, $(c, c + \delta) \subseteq Domn(f) \cap Domn(g)$, and both f, g are right differentiable at c.
Prove that $f + g$ is right differentiable at c and $(f + g)'_+(c) = f'_+(c) + g'_+(c)$.

Deduce that for each $\lambda, \mu \in \mathbb{R}$, $\lambda f + \mu g$ is right differentiable at c.

(b). Show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2|1 - x| + 3|1 - x^2|$ is right differentiable at 1 and find $f'_+(1)$.

3. (a). State and prove the Chain Rule.

(b). Let $h(x) = x\sqrt{x}, x \in (0, \infty)$. Show that h is differentiable on $(0, \infty)$.

(c). Let $f(x) = x^2 + 3x - 2$ and $g(x) = \sqrt{x + 1}$ for $x \in (0, \infty)$.

(i). Find $(f + g)', (f - g)'$ and $(fg)'$.

(ii). Show that $g(x) > 0$ for $x \in (0, \infty)$ and find $\left(\frac{f}{g}\right)'$.

4. (a). Let $a, b, c \in \mathbb{R}$ be such that $a^2 > 3b$. Let $f(x) = x^3 + ax^2 + bx + c$ for $x \in \mathbb{R}$.

Also let α, β be the roots of $3x^2 + 2ax + b = 0$ with $\alpha < \beta$.

Show that (i). $f(x) - f(\beta) = (x - \beta)^2\left(x - \alpha + \frac{\beta - \alpha}{2}\right)$

(ii). $f(x) - f(\alpha) = (x - \alpha)^2\left(x - \beta + \frac{\alpha - \beta}{2}\right)$ for each $x \in \mathbb{R}$.

Deduce that f has a local minimum at β and a local maximum at α.

(b). Suppose f is a function defined on an open interval (a, b) and suppose that f has a local maximum at $c \in (a, b)$. Prove that if f is differentiable at c, then $f'(c) = 0$.

5. (a). State Rolle's theorem.

Let f be three times differentiable on (a, b) and let f and f' be continuous on $[a, b]$.
Suppose that $f(a) = f'(a) = f(b) = f'(b) = 0$.
Show that there exists $c \in (a, b)$ such that $f'''(c) = 0$.

(b). State the Mean-Value theorem.

(i). Let I be an open interval. Suppose that $f(x)$ is a function such that $f'(x) = 0$ for each $x \in I$.
Prove that $f(x)$ is constant on I.

(ii). Let $f(x)$ be a function differentiable on $\mathbb{R}$ such that $f'(x) = 3x^2 + 2x + 1$ for $x \in \mathbb{R}$.
Prove that there exists $c \in \mathbb{R}$ such that $f(x) = x^3 + x^2 + x + c$ for $x \in \mathbb{R}$.

6. (a). State the Intermediate – Value Property for the Derivative.

Let $g(x) = \begin{cases} 1 & ; x \in \mathbb{Q}, \\ -1 & ; x \in \mathbb{Q}^c. \end{cases}$

Prove that there does not exist a function f defined on $\mathbb{R}$ such that $f'(x) = g(x)$ for $x \in \mathbb{R}$.

(b). State Taylor's theorem.

Let $f(x) = \log(1 + x)$.

(i). Calculate the fourth order Taylor polynomial $P_4(x)$ for $f(x)$ at $x = 0$.

(ii). Write down a formula for the fourth remainder term $R_4(x)$, and deduce that $\frac{x^5}{5(1+x)^5} \leq f(x) - P_4(x) \leq \frac{x^5}{5}$ for all $x > 0$.

...End...