(16)
00274

The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination - 2020/2021
Pure Mathematics - Level 04
PEU4300/PUU2140 –Real Analysis I

Date: 08.12.2021 **Time 1.30 p.m.-3.30 p.m.**

Answer Four questions only

Question (01)

(a) Using $\varepsilon - N$ definition, prove each of the following:

(i) $\lim_{n\to\infty} \frac{n-2}{n+5} = 1$,

(ii) $\lim_{n\to\infty} \frac{n+2}{\left(n-\frac{11}{2}\right)} = 1$.

(b) (i) Let r be a real number such that $0 < r < 1$ and let $m \in \mathbb{N}$. Prove that $0 < r^m < \frac{r}{m(1-r)}$. Deduce that $\lim_{n\to\infty} r^n = 0$.

(ii) Let r be a real number such that $-1 < r < 0$. Prove that $\lim_{n\to\infty} r^n = 0$.

(iii) Find all values of $r \in \mathbb{R}$ such that $\left\langle \frac{r^n}{1+r^n} \right\rangle$ converges. Carefully justify your answer.

Question (02)

(a) Prove or disprove each of the following

(i) There is a strictly increasing sequence $\langle a_n \rangle$ such that $\{a_n : n \in \mathbb{N}\} = \mathbb{N}$.

(ii) There is a strictly decreasing sequence $\langle a_n \rangle$ such that $\{a_n : n \in \mathbb{N}\} = \mathbb{N}$.

(b) Write down two bounded sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ such that for each $n \in \mathbb{N}$, $x_n < y_n$ and $\limsup x_n = \limsup y_n = 1$, $\liminf x_n = \liminf y_n = 0$.

(c) Write down two sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ such that lim sup$x_n = 3$, lim sup $y_n = 5$ and lim sup $x_n y_n = 4$.

(d) Let $\langle x_n \rangle$ be the sequence such that, $x_n = \sin\left(\frac{n\pi}{4}\right) + \cos\left(\frac{n\pi}{2}\right)$. Find lim inf x_n. Carefully prove your answer.

Question (03)

(i) Prove that every Cauchy sequence is bounded.

(ii) Consider the Fibonacci sequence $\langle x_n \rangle$ given by $x_1 = 1,\ x_2 = 1,\ x_{n+2} = x_n + x_{n+1}$ for each $n \in \mathbb{N}$. Using induction show that $x_{n+2}x_n - x_{n+1}^2 = (-1)^{n+1}$ for each $n \in \mathbb{N}$.

Let $a_n = \frac{x_{n+1}}{x_n}$ for each $n \in \mathbb{N}$. Show that $|a_n - a_{n+1}| < \frac{1}{n^2}$ for each $n > 4$.

Deduce that $\langle a_n \rangle$ is Cauchy. Prove that $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$.

Question (04)

(i) Prove that $\sum_{r=N+1}^{2N} \frac{1}{r} \geq \frac{1}{2}$, where $N \in \mathbb{N}$. Deduce that $\sum_{r=1}^{\infty} \frac{1}{r}$ is divergent.

(ii) Find the sum $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ in terms of n. Show that $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$ is convergent and find the sum.

(iii) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms. Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, where l is a non zero real number. Prove that either both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent or else both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent.

Question (05)

Determine the convergence or divergence of each of the following series.

(i) $\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)$

(ii) $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

(iii) $\sum_{n=1}^{\infty}\frac{(-1)^n(101)^{n+100}}{n(100)^{n+101}}$

(iv) $\sum_{n=1}^{\infty}\frac{1}{(2^n+(-2)^n+1)}$

(v) $\sum_{n=1}^{\infty}\frac{(-1)^n\left(sin\left(\frac{1}{n}\right)+cos\left(\frac{1}{n}\right)\right)}{n}$

Question (06)

(a) Define the radius of convergence of the power series $\sum_{n=0}^{\infty}a_n\,x^n$

Determine the radius of convergence of each of the following power series.

(i) $\sum_{n=1}^{\infty}(3^n+5^n)\,x^n$, (ii) $\sum_{n=1}^{\infty}\left(\frac{1}{4+(-1)^n}\right)^n x^n$

(b) Find whether each of the following series is conditionally convergent, absolutely convergent or divergent.

(i) $\sum_{n=1}^{\infty}\frac{1+(-1)^n n^2}{n^3}$, (ii) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{n^2+1}$, (iii) $\sum_{n=1}^{\infty}\frac{1+(-1)^n n^2}{n^4}$