The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	: Mathematics
Level	: Five (05)
Name of the Examination	: Final Examination
Course Code and Title	**: PEU5300 – Riemann Integration**
Academic Year	: 2020/2021
Date	: 12.12.2021
Time	: 09.30 a.m. – 11.30 a.m.
Duration	: 2 hours
Index number	:

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **Six (06)** questions in **Two (02)** pages.
3. Answer any **Four (04)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary
5. Relevant log tables are provided where necessary.
6. Having any unauthorized documents/ mobile phones in your possession is a punishable offense
7. Use blue or black ink to answer the questions.
8. Circle the number of the questions you answered in the front cover of your answer script.
9. Clearly state your index number in your answer script

Q1)

Let f be a bounded function on $[a, b]$. Define the lower integral and upper integral of f on $[a, b]$.

Let $f(x) = \begin{cases} 3, & x \in [1,5] \cap \mathbb{Q} \\ 2+x, & x \in [1,5] \cap \mathbb{Q}^c. \end{cases}$ Show that $\overline{\int}_1^5 f(x) = 20$ and $\underline{\int}_1^5 f(x) = 12$.

Q2)

(a) Let f be a bounded function on $[a, b]$. Prove that

$$\underline{\int}_a^b f(x)dx \leq \overline{\int}_a^b f(x)dx.$$

(b) Let f be a continuous function on $[0, 1]$ such that $f(x) \geq 0$ for all $x \in [0,1]$. Show that $\int_0^1 f(x)dx > 0$ if and only if there is a point $c \in [0,1]$ at which $f(c) > 0$.

Q3)

State the Riemann's Criterion for integrability of a bounded function f defined on $[a, b]$.

Using the Riemann's Criterion prove that,

(i) if f is a bounded and monotonically increasing function on $[a, b]$, then f is Riemann integrable on $[a, b]$, and

(ii) if f is Riemann integrable on $[a, b]$, then $|f|$ is also Riemann integrable on $[a, b]$.

Show that,

(iii) by using (i) above, the function f defined by $f(x) = x^{8/3} - 4x^{2/3}$ is Riemann integrable on $[2, 3]$, and

(iv) by giving an example, the converse of (ii) above is not necessarily true.

Q4)

(a) Let f be a Riemann integrable function on $[a, b]$ such that $|f(x)| \leq M$ for each $x \in [a, b]$.

Clearly stating any results you use, show that,

(i) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx \leq M(b-a)$ and

(ii) $\left|\int_{-2\pi}^{2\pi} x^2 \cos^{10}(e^{x+1})dx\right| \leq \frac{16}{3}\pi^3$.

(b) Let f be a continuous function $\mathbb{R}$ and define $F(x) = \int_0^{\sin x} f(t)dt$ for $x \in \mathbb{R}$. Show that F is differentiable on $\mathbb{R}$ and compute F'.

If in the above $f(t) = \sqrt{1+t^2}$ and $G(y) = \int_0^y F(x)dx$, then find $G''(\pi/6)$.

Q5)

(a) State the Mean Value Theorem for integrals.

Let $f(x) = \begin{cases} x, & x \in [0.2] \setminus \{1\} \\ 0, & x = 1. \end{cases}$

Prove that f is Riemann integrable on $[0, 2]$ and for each $\alpha \in [0, 2]$, $\int_0^2 f(x)dx \neq f(\alpha)(2-0)$.

Why does the Mean Value Theorem not applicable to f? Justify your answer.

(b) Using the properties of Riemann Sums, find two different integrals over the intervals $[0, 3]$ and $[-1, 2]$ that are represented by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right)\left[4 - \frac{9i^2}{n^2}\right].$$

Hence find the above limit.

Q6)

Consider the following integrals:

(i) $\int_0^2 \frac{1}{\sqrt{2-x}}dx$, (ii) $\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$,

(iii) $\int_{-\infty}^{-2} \frac{x^4+x^2}{x^6+10}dx$, (iv) $\int_4^5 \frac{x-2}{\sqrt{x}(x-4)}dx$.

(a) Explain why each of the above is an improper integral.
(b) Determine the convergence of each of them. Justify your answers.

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