The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2020/2021
Pure Mathematics - Level 05
PEU5305 – Complex Analysis I

Duration: - Two hours

Date: - 19-12-2021 **Time: - 01.30 p.m. – 3.30 p.m.**

Answer **four** questions **only**

Question 01

(a) Give the definition of each of the following:

(i) A complex- valued function $f(z)$ is analytic in an open set of $\mathbb{C}$.

(ii) A complex- valued function $f(z)$ is analytic at a point $z_0 \in \mathbb{C}$.

(b) Prove or disprove:

If a complex value function $f(z)$ is differentiable at a point $z_0 \in \mathbb{C}$, then it is analytic at z_0.

(c) Prove that the function $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic in $\mathbb{C}$.

(d) Can you find a function $v(x, y)$ such that $2xy + 3xy^2 - 2y^3 + iv(x, y)$ is analytic in $\mathbb{C}$? Justify your answer.

Question 02

(i) Prove that $\cos z$ is not bounded in $\mathbb{C}$.

(ii) Solve the equation $\cos z = 2$.

(iii) Find the value of i^i.

(iv) Find the value of $\mathrm{Log}(-1-i)$.

Question 03

(a) Let $f(z) = \frac{1}{(z+1)}$, $g(z) = \frac{1}{(z+3)}$ and $h(z) = \frac{1}{(z+1)(z+3)}$.
Find the Laurent series of

(i) $f(z)$ when $|z| > 1$,

(ii) $g(z)$ when $|z| < 3$,

(iii) $h(z)$ when $1 < |z| < 3$.

(b) Find the Laurent series about the indicated singularity for each of the following functions. Classify the indicated singularity of each function as a pole (with order), removable singularity or an essential singularity.

(i) $\frac{z-\sin z}{z^3}$; $z = 0$,

(ii) $\frac{z}{(z+1)(z+2)}$; $z = -2$,

(iii) $(z-3)\sin\frac{1}{z+2}$; $z = -2$.

Question 04

(a) Evaluate each of the integrals $\int_C |z|^2 dz$ and $\int_C \frac{1}{z^2} dz$, where the contour C is

(i) the line segment with initial point -1 and final point i,

(ii) the arc of unit circle Im z ≥ 0 with initial point -1 and final point i.

(b) (i) State Cauchy's Theorem.

(ii) Let C_1 and C_2 be two simple closed contours such that C_2 lying entirely inside C_1. Show that if $f(z)$ is analytic in a set $G \subset \mathbb{C}$ consisting of points on and between C_1 and C_2, then $\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$.

(c) Evaluate $\int_C \frac{1}{4z^2+1} dz$, where C is

(i) the circle $|2z - i| = \frac{1}{2}$, oriented counterclockwise,

(ii) the circle $|2z + i| = \frac{1}{2}$, oriented counterclockwise,

(iii) the square with vertices $\pm 1 \pm i$ oriented counterclockwise.

Question 05

(a) State Cauchy's Integral Formula.

Using Cauchy's Integral Formula, evaluate the integral $\int_C \frac{\cos z}{z(z^2+8)} dz$, where C is the circle with radius 3/2 centered at 0, oriented counterclockwise.

(b) Apply Cauchy's integral formula to cosz with the contour $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, to show that $\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta) d\theta = 2\pi$.

Question 06

(a) Use the residue theorem to calculate each of the following integrals:

(i) $\int_C \frac{5z-2}{z(z-1)}dz$, where C is the circle $|z| = 2$, oriented counterclockwise.

(ii) $\int_C z^2 \sin\left(\frac{1}{z}\right)dz$, where C is the circle $|z| = 1$, oriented counterclockwise.

(b) Use the residue theorem to show that $\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \frac{2\pi}{\sqrt{5}}$.