00083

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	: **Mathematics**
Level	: **5**
Name of the Examination	: **Final Examination**
Course Code and Title	: **PUU5301– RING THEORY**
Academic Year	: **2020/ 2021**
Date	: **11 -03 - 2022**
Time	: **09.30 a.m. – 11.30 a.m.**
Duration	: **Two Hours**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of **six (06)** questions in **two (02)** pages.
3. Answer any **four (04)** questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary
5. Relevant log tables are provided where necessary.
6. Having any unauthorized documents/ mobile phones in your possession is a punishable offense,
7. Use blue or black ink to answer the questions.
8. Circle the number of the questions you answered in the front cover of your answer script.
9. Clearly state your index number in your answer script,

Q1) Define a new addition $\oplus$ and multiplication $\odot$ on $\mathbb{Z}$ in terms of ordinary addition, subtraction, and multiplication by

$$a \oplus b = a + b - 1, \text{ and } a \odot b = a + b - ab.$$

(i) Prove that $(\mathbb{Z}, \oplus, \odot)$ is a commutative ring.
(ii) Does $(\mathbb{Z}, \oplus, \odot)$ have a multiplicative identity? Justify your answer.

Q2) Consider the subset $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ of the real numbers $\mathbb{R}$. Assuming $\mathbb{R}$ is a ring with respect to the ordinary addition and multiplication of $\mathbb{R}$, prove that $\mathbb{Z}[\sqrt{2}]$ is an integral domain but not a field.

Q3)

(a) Find the $\gcd(1064, 856)$ and a pair of integers (m, n) such that $1064m + 856n = \gcd(1064, 856)$.

(a) Let $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$. Find all the units of the ring $\mathbb{Z}[i]$.

Q4)

(a) Let R, S be commutative rings, and $\varphi : R \to S$ be a ring homomorphism. Prove that if J is an ideal of S, then the pre-image $I = \varphi^{-1}(J)$ is an ideal of R.

(b) Let $C(\mathbb{R}) = \{f : f \text{ is a continuous function from } \mathbb{R} \text{ to } \mathbb{R}\}$. Assuming $C(\mathbb{R})$ is a commutative ring under usual addition and multiplication of functions, prove that $I = \{f \in C(\mathbb{R}) : f(1) = 0\}$ is an ideal but that $J = \{f \in C(\mathbb{R}) : f(1) \in \mathbb{Z}\}$ is not an ideal.

Q5)

(a) Let R be a commutative ring with unity, I and J be ideals of R. Prove that the subsets $I \cap J, IJ$, and $I + J$, defined in the usual notation, are ideals of R.

(b) Consider the two ideals $I = 225\mathbb{Z}$ and $J = 30\mathbb{Z}$ in the ring $\mathbb{Z}$. Determine the ideals $I \cap J, IJ$, and $I + J$.

Q6)

(a) Let R be an integral domain. Define what is meant by,
(i) R is a Euclidean domain, and
(ii) R is a principal ideal domain.

(b) Prove that every Euclidean domain is a principal ideal domain.

(c) Prove that the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients is not a principal ideal domain.

=========//=========