00135

The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc/ B. Ed Degree Programme

Department	: Mathematics
Level	: 05
Name of the Examination	: Final Examination
Course Code and Title	: PEU5303- Number Theory
Academic Year	: 2020/2021
Date	: 18.03.2022
Time	: 2.00 p.m.-4.00 p.m.
Duration	: 2 Hours

General Instructions

1. Read all instructions carefully before answering the questions.
2. This question paper consists of 06 questions in 03 pages.
3. Answer any 04 questions only. All questions carry equal marks.
4. Answer for each question should commence from a new page.
5. Draw fully labelled diagrams where necessary
5. Relevant log tables are provided where necessary.
6. Having any unauthorized documents/ mobile phones in your possession is a punishable offense
7. Use blue or black ink to answer the questions.
8. Circle the number of the questions you answered in the front cover of your answer script.
9. Clearly state your index number in your answer script

Question (01)

(i) Prove that $\sqrt{\frac{2}{3}}$ is irrational.

(ii) Find all positive integer solutions of the equation $x^2 + y^2 = 233$. Justify your answer.

(iii) Let p be odd prime. Find all positive integer solutions of the equation $x^2 - y^2 = p^3$

(iv) Prove that there are infinitely many positive integers n for which the equation, $x^2 - y^2 = n$ has no integer solutions.

(v) Prove that there are infinitely many positive integers n for which the equation, $x^2 + y^2 = n$ has no integer solutions

Question (02)

(a) Prove or disprove the following;

(i) There exist $a, b \in \mathbb{Q}$ such that $a > 0,\ b > 0$ and $a^2 + b^2 = 1$.

(ii) There exist $a, b \in \mathbb{Q}$ such that $a > 0,\ b > 0$ and $a^2 + b^2 = 3$.

(b) Use Euclidean Algorithm to obtain integers x and y such that $\gcd(119, 272) = 119x + 272y$.

(c) Find all integers that give the remainders 1, 2 and 3 when divided by 3, 4 and 5 respectively. (Hint: use Chinese Remainder Theorem)

Question (03)

(a) (i) Prove the Euclid's Lemma. I.e., prove that if $a|bc$ then $a|c$, for any integers a, b, c with $\gcd(a, b) = 1$.

(ii) Deduce that, for $a, b \in \mathbb{Z}$ and for prime number p, if $p|ab$ then $p|a$ or $p|b$.

(iii) Prove that $\sqrt{\frac{3}{5}}$ is irrational.

(b) Find all integer solutions of the system of equations,

$5x + 3y + \frac{1}{3}z = 100,\ x + y + z = 100.$

(Hint: convert the system to linear Diophantine equation of two variables)

Hence, determine all positive integer solutions to above system.

Question (04)

(i) Let p, q be twin primes such that $p - q = 2$ and $q > 3$. Prove that $pq + 1$ is a square number and it is divisible by 36.

(ii) Prove that a number of the form $21k + 5$, where $k \in \mathbb{N}$, is not a twin prime.

(iii) The middle number of the twin primes 3 and 5 is 4 and which is a square number. Are there any other twin primes with the number in middle is a square? Justify your answer.

(iv) Prove that the sequence $(6n^2 + 16n + 1)_{n\in\mathbb{N}}$ has infinitely many composite numbers.

Question (05)

Let $R_n = \underbrace{111 \ldots 11}_{n \text{ digits}}$ be the number with n occurrences of 1 in its decimal form.

Prove the following.

(i) $R_n = \frac{10^n - 1}{9}$ for each $n \in \mathbb{N}$.

(ii) R_n is divisible by 9 iff n is divisible by 9.

(iii) R_n is divisible by 7 iff n is divisible by 6.

(iv) If R_n is prime then n is prime.

(v) for $n \geq 2$, R_n not a square number.

Question (06)

(a) (i) Let a be a positive integer and p be a prime number with $\gcd(a, p) = 1$. Prove that $\{0, a, 2a, 3a, \ldots (p-2)a, \ (p-1)a\}$ is a complete set of residues modulo p. Hence, deduce that $a^{p-1} \equiv 1 (\text{mod } p)$.

(ii) Prove that $n^7 - n$ is divisible by 42 for any integer n.

(b) (i) Prove that if p is odd prime and m is an integer such that $1 < m < p - 1$, then $m^2 \not\equiv 1 \ (\text{mod } p)$. Deduce that $(p-1)! \equiv -1 \ (\text{mod } p)$

(ii) Prove that $18! \equiv -1 (\text{mod } 437)$