The Open University of Sri Lanka

Faculty of Natural Sciences

B.Sc/B. Ed Degree Programme

Department	**: Mathematics**
Level	**: 05**
Name of the Examination	**: Final Examination**
Course Code and Title	**: PEU5304 - Introduction to Complex Analysis**
Academic Year	**: 2021/2022**
Date	**: 10.10.2022**
Time	**: 1.30 pm– 3.30 pm**
Duration	**: 02 hrs**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This paper consists of **TWO** sections, Section **A** and Section **B**. Section **A** is compulsory and it consists of **SIX** Structured Essay Questions. You should answer in the space under each question.
3. Section **B** consists of **FIVE** essay type questions and answers only **THREE** of them. Each question in Section **B** carries 100 marks.
4. This paper consists of **05** pages.
5. Answer for each question in section **B** should commence from a new page.
6. Involvement in any activity that is considered as an exam offence will lead to punishment.
7. Use a blue or black ink pen to answer the questions.
8. Clearly state your index number in your answer script.
9. At the end of the exam, attach Section **A** to the answer booklet and hand over to the supervisor.

SECTION A

1. Answer all the questions in the space provided.

   Prove or disprove each of the following statements. Justify your answer.

   a) $\left|z_1\overline{z_2} - z_2\overline{z_1}\right| \leq 2\left|z_1 z_2\right|$, where $z_1, z_2 \in \mathbb{C}$.

   b) $\left|z_1 + z_2\right|^2 = \left|z_1\right|^2 + 2\mathrm{Re}\left(z_1\overline{z_2}\right) + \left|z_2\right|^2$, where $z_1, z_2 \in \mathbb{C}$.

   c) If z is a point in the third quadrant, then $i\overline{z}$ is in the fourth quadrant.

d) If z be a complex number such that $|z|=3$, then $\left|\frac{z^2}{z^4+5z^2+4}\right| \leq \frac{9}{40}$.

e) $\text{Arg}\left(-\frac{3\pi}{2}\right)=-\frac{\pi}{2}$.

f) If a and b be two complex numbers such that $|a|<1$ and $|b|<1$, then $\left|\frac{a-b}{1-\bar{a}b}\right|<1$.

SECTION B

Answer **THREE** Questions **ONLY**.

2.

a) Let z_1, z_2 be two non-zero complex numbers. Prove each of the followings:

i. $\frac{z_1}{z_2}$ is a non-zero complex number.

ii. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

b)

i. Find all the complex numbers z such that $\arg(z+1) = 2n\pi + \frac{\pi}{4}$ and

$\arg(z-1) = 2n\pi + \frac{3\pi}{4}; n \in \mathbb{Z}$.

ii. Suppose z_1, z_2 are distinct non-zero complex numbers such that

$\arg\left(\frac{z_1+z_2}{z_1-z_2}\right) = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$ or $\arg\left(\frac{z_1+z_2}{z_1-z_2}\right) = -\frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$.

Prove that $|z_1| = |z_2|$.

3.

a) Find and represent the loci of the points in the complex plane that represent the complex numbers z such that;

i. $\mathrm{Re}\left(\frac{z-6i}{z+4i}\right) = 0; z \neq -4i$,

ii. $\mathrm{Arg}\, z = -\frac{\pi}{4}$.

b) If $\arg\left(\frac{z-2}{z-6i}\right) = \frac{\pi}{2}$, then show that z lies on the circle $(x-1)^2 + (y-3)^2 = 10$ for each $x, y \in \mathbb{R}$.

c) Let $z_1 = 1+i$ and $z_2 = 1+\sqrt{3}i$. Construct the points p_1 and p_2 in the z - plane hat represent the complex numbers z_1 and z_2 respectively. Use the point p_1 and p_2 to construct the point p that represents the complex number $\frac{z_1}{z_2}$.

4.

a) State the **De Moivre's Theorem** for a positive integer n. Deduce that the theorem is true when integer n is a negative integer.

b) Using De Moivre's Theorem, show that $\frac{\cos 4\theta}{\cos^4\theta} = 1-6\tan^2\theta+\tan^4\theta$ for $\theta \in \mathbb{R} - \left\{\frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\right\}$.

Hence, obtain the value of $\tan\left(\frac{\pi}{8}\right)$.

c) Express $\frac{1}{2}\left(\sqrt{3}-i\right)$ in the form of $\cos\theta+i\sin\theta$, where $\theta \in \mathbb{R}$ and **hence** find the value of $\left(\frac{\sqrt{3}-i}{2}\right)^{2021}$.

5.

a) Let n be a positive integer greater than 1. Then prove that $\cos\left(\frac{2\pi}{n}\right)+i\sin\left(\frac{2\pi}{n}\right)$ is a primitive n^{th} root of unity.

b) Find the fifth roots of -1 (leave the answers in the form of $\cos\theta+i\sin\theta; -\pi < \theta \le \pi$) and **hence**, by considering the solutions of the equation $z^5+1=0$, prove that $\cos\frac{3\pi}{2}+\cos\frac{\pi}{5}=\frac{1}{2}$. (**Hint**: Consider the summation of roots.)

c) State The Remainder Theorem for the complex polynomial $P(z)$. Consider the complex cubic equation $P(z) = 2z^3-10z-5+\left(2z^2-5z+14\right)i = 0$. Verify that i is a root of $P(z)=0$. Find all the roots of $P(z)=0$.

6.

a) Find all the roots of the equation $\log z = \left(\frac{5\pi}{2}\right)i$.

b) Prove or disprove each of the following statements:

i. For each complex number z, $\log e^z = z + i2n\pi$, $n \in \mathbb{Z}$.

ii. Let z_1, z_2 be arbitrary non-zero complex numbers then,

$$\mathrm{Log}\left(\frac{z_1}{z_2}\right) = \mathrm{Log}z_1 - \mathrm{Log}z_2.$$

iii. $\log i^{\frac{1}{2}} = \frac{\log i}{2}$.

c) Show that for each $z \in \mathbb{C}\setminus\{i,-i\}$, $\tan^{-1} z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$. Evaluate $\tan^{-1}(1)$.