The Open University of Sri Lanka
B.Sc./ B.Ed. Degree Programme
Final Examination-2015/2016
Pure Mathematics- Level-05
PUU 3244 - Number Theory & Polynomials

Duration: Three Hours.

Date: 08-01-2017 **Time: 1.30p.m. To 4.30p.m.**

Answer Five questions only.
(State clearly any result that you used, without proof.)

(1) (a) Prove each of the following:

(i) If $n \in \mathbb{N}$ and $n > 1$ then $n-1 \in \mathbb{N}$.

(ii) There is no $n \in \mathbb{N}$ such that $0 < n < 1$.

(b) State and prove the Mathematical Induction.

(c) Using mathematical induction, prove that $(x+y)^n = \sum_{r=0}^{n} {}^nc_r x^{n-r} y^r$ for all positive integer n; where ${}^nc_r = \frac{n!}{(n-r)!r!}$.

(2) (a) Define each of the following:
(i) A prime number.
(ii) Division Algorithm.

(b) If $a, b \in \mathbb{Z}$ and $ab = 1$, prove that $|a| = 1$ and $|b| = 1$.

(c) Prove that $2^{2n}+1$ is divisible by 5 when n is an odd natural number.

(d) (i) Find the greatest common divisor g of 7469 and 2464, and hence find integers x and y such that $7469x + 2464y = g$.

(ii) Find the least common multiple of 4999 and 1109.

(3) Prove that every positive integer $n > 1$ can be expressed uniquely as a product of prime numbers except for the order in which the prime factors occur.

(4) (a) Prove that for a given $n \in \mathbb{Z}$ there exist a unique $r \in \mathbb{Z}_m$ such that $n \equiv r \pmod{m}$.

(b) Find the integer in $\mathbb{Z}_9$ which is congruent to $4 \times 8 \times 14 \times 403 \times 22$ modulo 9.

(c) Prove that $2^{12} \equiv 1 \pmod{63}$ and deduce that

$2^{300} \equiv 1 \pmod{63}$ and $2^{12} \equiv 1 \pmod{819}$.

(5) (a) Let R be a commutative ring. If $f(x), g(x) \in R[x]$ and $g(x)$ is monic, then prove that there exists a unique $q(x), r(x) \in R[x]$ such that $f(x) = q(x)g(x) + r(x)$ with $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

(b) If $f(x) = x^4 + 4x^3 + 3x^2 + x + 1$ and $g(x) = 2x^3 + x^2 - x - 3$ are polynomials over $\mathbb{Z}_5[x]$ then find the greatest common divisor d of $f(x)$ and $g(x)$ and express it in the form $d = fu + gv$ where $u, v \in \mathbb{Z}_5[x]$.

(6) (a) State and prove Eisentein's irreducibility criteria.

(b) Determine whether each of the following polynomials are irreducible over $\mathbb{Q}[x]$.

(i) $f(x) = 8x^3 + 6x^2 + 9x + 24$.

(ii) $f(x) = x^3 + 3x^2 - 5$.

(7) (a) (i) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and $n \geq 1$. If $\alpha \in \mathbb{Q}$ is a zero of $f(x)$ and $\alpha = \frac{r}{s}$ such that $(r, s) = 1$, then show that $r \mid a_0$ and $s \mid a_n$.

(ii) Find all rational roots of the polynomial $6x^4 - 25x^3 + 35x^2 - 20x + 4$ over $\mathbb{Q}$.

(b) (i) State and prove the Remainder Theorem.

(ii) Find all prime numbers p such that $x - 2$ is a factor of $f(x) = x^4 + 3x^3 + 2x^2 + x + 1$ in $\mathbb{Z}_p[x]$.

(8) (a) Let $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x]$, $a_n \neq 0$ and $\alpha_1, \alpha_2 \ldots\ldots \alpha_n$ are the zeros of $f(x)$ in $\mathbb{C}$.

Show that (i) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_0 S_{m-n} = 0$, if $m > n$,

(ii) $a_n S_m + a_{n-1} S_{m-1} + \ldots\ldots + a_{n-m+1} S_1 + m a_{n-m} = 0$, if $m \leq n$,

where $S_r = \sum_{i=0}^{n} \alpha_i^r$.

(b) If $a, b, c \in \mathbb{C}$ such that $a + b + c = 0$, then prove that,

$4(a^7 + b^7 + c^7) = 7abc(a^2 + b^2 + c^2)^2$.