THE OPEN UNIVERSITY OF SRI LANKA

Final Examination 2015/2016

Bachelor of Technology Honours in Engineering (Level 5)

MPJ5134 - The Nature of Mathematics I

Duration: Three Hours

Date: 22-11-2016 Time : 1330 hrs - 1630 hrs

Instructions

- Answer Five (05) questions only
- This paper contains five(5) pages
- State clearly any assumptions you required
- All symbols are in standard notations unless they are defined

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Some important notes:

- Axioms for real numbers

For arbitrary real numbers a, b, c:

A0. $a + b$ and $a.b$ are real numbers

A1. $a + b = b + a$

A2. $a + (b + c) = (a + b) + c$

A3. $a + 0 = a$

A4. $a + (-a) = 0$

M1. $ab = ba$

M2. $a(bc) = (ab)c$

M3. $a.1 = a$

M4. $a.\frac{1}{a} = 1$, if $a \neq 0$

D. $a(b + c) = ab + ac$

- Order Axioms

  P1. If a, b in $\mathbb{P}$, then $a+b$ in $\mathbb{P}$

  P2. If a, b in $\mathbb{P}$, then ab in $\mathbb{P}$

  P3. For any real numbers a, b of $\mathbb{R}$, one and only one of the following is true: a in $\mathbb{P}$, $a = 0, -a$ in $\mathbb{P}$.

- Integer a divide $b(a|b)$ if and only if $b = ca$ for some integer c .

- Mathematical Induction:

  If we can prove both of the following steps, we then know that the proposition is a result (Theorem), true for all n

  (I) The initial step:

  Prove the proposition is true for $n = 1$

  (II) Inductive step:

  Assume the proposition is true for $n = k$, and on this basis prove it is true for $n = k + 1$

End of the note

1. (a) Let $a, b, c \in \mathbb{R}$. Using the axioms of real numbers prove that:

   i. $a + b = a + c \Rightarrow b = c$.

   ii. $a.0 = a.0 + a.0$.

   iii. $a + (-1).a = a.0$. [30%]

   (b) i. Using the axiom (A3),(a) (ii) and (a) (i), prove that $a.0 = 0$.

   ii. Using (a) (iii), (b) (i), axioms (A4) and (A1),and (a) (i), prove that $(-1).(-1) = 1$.

   iii. Using (a) (iii), (b) (i), axioms (A4) and (A1),and (a) (i), prove that $(-1).(-1) = -(-1)$. [30%]

   (c) Let $\alpha, a, b, \theta \in \mathbb{R}$. Assuming the result "$|x| \leq \alpha \Leftrightarrow -\alpha \leq x \leq \alpha$", prove that if $a \leq x \leq b$ and $a \leq y \leq b$, then $|x - y| \leq b - a$. Deduce that $||\sin\theta| - |\cos\theta|| \leq 1$. [20%]

(d) Let n be a positive integer, prove $n! < 2^n$ by Mathematical Induction. [20%]

2. (a) Use truth tables to prove that the following propositions are logically equivalent:

i. $\sim (p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$.

ii. $p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$. [30%]

(b) Determine the tautology and contradiction of the following propositions:

i. $(\sim q \Rightarrow \sim p) \Rightarrow (p \Rightarrow q)$.

ii. $\sim [((p \wedge (p \Leftrightarrow q)) \Rightarrow q]$. [30%]

(c) Consider the argument: If I study Logic, then I will get A pass. If I do not go to party, then I will study Logic. But I did not get A pass. Therefore, I went to party.

i. Express the above argument in symbolic form with the following notations.

- p: I study Logic.
- q: I will get A pass.
- r: I go to party.

ii. Determine the validity of the above argument. [20%]

(d) Determine whether the collection of propositions $p \Rightarrow q, \sim p \vee \sim q$, $\sim (q \Rightarrow p)$ are consistent or inconsistent. [20%]

3. (a) Give a direct proof for each the following.

i. If a is an even integer and b is an odd integer, then $ab + 7$ is an odd integer.

ii. Suppose $m \in \mathbb{Z}$. If 4 divides $(m^2 - 1)$, then m is odd.

iii. Let n be an even number. Show that $1 + (-1)^n(2n - 1)$ is divisible by 4. Deduce that if n is an odd number, then 4 divides $1 + (-1)^n(2n - 1)$.

[50%]

(b) Show by using proof by contradiction that:

i. Let a and b be integers. If $a^2 + b^2 = 9$, then either a or b is even.

ii. Prove that there is no rational number whose square is 2.

iii. Let x be real number. If $x > 0$, then $x + \frac{1}{x} \geq 2$. [50%]

4. (a) Suppose A, B and C are sets. Show by method of Venn diagram that:

i. $A \backslash (B \cup C) = (A \backslash B) \cup (A \backslash C)$.

ii. $(A \backslash B) \cup (B \backslash A) = A \cup B$ if and only if $A \cap B = \emptyset$

iii. $(A \backslash B)^c = B \cup A^c$ [60%]

(b) Give an example to show that

i. $A \backslash (A \backslash B) \neq B$.

ii. $(A \backslash B) \cup (B \backslash C) = (A \backslash C)$ [40%]

5. (a) Suppose, E is a universal set. Now consider the following nine results for any sets A, B, C of E.

R1: $A \cap \emptyset = \emptyset$

R2: $A \cup \emptyset = A$

R3: $A \cap A^c = \emptyset$

R4: $A \cup A^c = E$

R5: $A \cap E = A$

R6: $A \cap A = A$

R7: $A \cap B = B \cap A$

R8: $(A \cup B)^c = A^c \cap B^c$

R9: $(A \cap B)^c = A^c \cup B^c$

R10: $A \backslash B = A \cap B^c$

R11: $A \cap (B \cap C) = (A \cap B) \cap C$

R12: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Now, using only nine results and the properties of "=", prove algebraically the following for arbitrary subsets A, B, C of E.

i. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$

ii. $A \cap (B \backslash C) = (A \cap B) \backslash (A \cap C)$ [50%]

(b) Prove that the results R3, R8 to R10 (in (a) above) by using logic method. [50%]

6. (a) Let $A = \{1, -1, 2\}$, $B = \{a, b\}$. Find each of the following set:

i. $A \times B$

ii. $B \times A$

iii. $A \times (B \times A)$

iv. $B^2 \times A$

v. $(A \times B)^2$ [50%]

(b) Let A, B, and C be sets . Prove that:

i. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

ii. $A \times (B \cup C) = (A \times B) \cup (A \times C)$ [50%]

7. (a) i. Define what is meant by integers a and b are relatively prime.

ii. Let a and b be integers. Show that if there exist integers x and y such that $ax + by = 1$, then a and b are relatively prime.

iii. Let a, b and c be integers such that $a|c$ and $b|c$. Show that if a and b are relatively prime, then $ab|c$. [50%]

(b) Using well ordering Principle of $\mathbb{N}$, show that:

i. $(\frac{\sqrt{2}+\sqrt{3}}{2})^n \leq \frac{\sqrt{2}^n + \sqrt{3}^n}{2}$ for all $n \in \mathbb{N}$.

ii. $\sum_r^n \frac{1}{r(r+1)} = 1 - \frac{1}{(n+1)}$ for all $n \in \mathbb{N}$. [50%]

End

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