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The Open University of Sri Lanka
Faculty of Natural Sciences
B.Sc. / B. Ed. Degree Programme

Department	**: Mathematics**
Level	**: 05**
Name of the Examination	**: Final Examination**
Course Code and - Title	**: PEU5304 - Introduction to Complex Analysis**
Academic Year	**: 2023/24**
Date	**: 04.10.2023**
Time	**: 1.30 p.m. To 3.30 p.m.**
Duration	**: Two Hours.**

General Instructions

1. Read all instructions carefully before answering the questions.
2. This paper consists of **TWO** sections, Section **A** and Section **B**. Section A is compulsory and it consists of **FIVE** Structured Essay Questions. You should answer in the space under each question.
3. Section **B** consists of **FIVE** essay type questions and answers only **THREE** of them. Each of the questions in Section **B** carries 100 marks.
4. This Paper consists of **06** pages.
5. Answer for each question should commence from a new page.
6. Involvement in any activity that is considered as an exam offense will lead to punishment.
7. Use blue or black ink to answer the questions.
8. Clearly state your index number in your answer script.
9. At the end of the exam, attach Section **A** to the answer booklet and hand over to the supervisor.

SECTION A

1. Answer all questions in the space provided.
   Prove each of the following statements.

   a) $||z_1| - |z_2|| \leq |z_1 - z_2|$, where $z_1, z_2 \in \mathbb{C}$.

   b) $|Re(2 + \bar{z} + z^3)| \leq 4$ when $|z| \leq 1$, where $z \in \mathbb{C}$.

   c) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,Re(z_1\bar{z}_2)$, where $z_1, z_2 \in \mathbb{C}$.

d) $Sin^{-1} z = -i \log [iz + (1 - z^2)^{\frac{1}{2}}]$, where $z \in \mathbb{C}$.

e) e^z is real if and only if $Im\ z = n\pi, n \in \mathbb{Z}$.

SECTION B

Answer **THREE** Questions **ONLY**.

2. a). Suppose z_1, z_2 are two distinct non-zero complex numbers. Prove each of the followings:

   i). If $|z_1 + z_2| = |z_1 - z_2|$, then $\arg(z_1) - \arg(z_2) = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$ or $\arg(z_1) - \arg(z_2) = -\frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$.

   ii). If $\arg\left(\frac{z_1+z_2}{z_1-z_2}\right) = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$ or $\arg\left(\frac{z_1+z_2}{z_1-z_2}\right) = -\frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$, then $|z_1| = |z_2|$.

   b). Find and represent the loci of the point in the complex plane that represent the complex numbers z such that;

   i). $Re\left(\frac{z-6i}{z+4i}\right) = 0; z \neq -4i$,

   ii). $Im\left(\frac{z-6i}{z+4i}\right) = 0; z \neq -4i$.

3. a). i). Let $p + qi$ be a root of $a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_n = 0$, where $a_1, a_2, a_3, \ldots, a_n, p$ and q are real and $a_0 \neq 0$. Prove that $p - qi$ is also a root of the above equation.

   ii). Consider the complex quartic equation $p(z) = z^4 - 4z^3 + 3z^2 + 14z + 26 = 0$

   Verify $-1 - i$ is a root of $p(z) = 0$. Find all the roots of $p(z) = 0$.

   b). If z_1, z_2, z_3 are the vertices of an equilateral triangle, show that,

   $$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1 .$$

   If $z_1 = 1 + i$ and $z_2 = 1 - i$, determine the two possible values of z_3 .

4. a). State De Moivre's theorem. Prove the theorem for all non-negative integers. Deduce that, the result is true for any negative integer.

   b). Express $\frac{1}{\sqrt{2}}(1-i)$ in the form of $cos\theta + i\, sin\theta$, where $\theta \in \mathbb{R}$. Hence find the value of $(\frac{1-i}{\sqrt{2}})^{-31}$.

   c). Let θ be a real number such that $\theta \neq 2k\pi$ for each integer k.

   Prove that $\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta = \frac{\sin(\frac{n+1}{2}\theta)\sin(\frac{n\theta}{2})}{Sin(\frac{\theta}{2})}$.

5. a). The all n distinct roots of $r_0(cos\theta_0 + isin\theta_0)$ is denoted by $[r_0(cos\theta_0 + isin\theta_0)]^{\frac{1}{n}}$, where n is a positive integer and $r_0 > 0$. Prove that

   $$[r_0(cos\theta_0 + isin\theta_0)]^{\frac{1}{n}} = \sqrt[n]{r_0}\left[\cos\left(\frac{\theta_0+2k\pi)}{n}\right) + i\ \sin\left(\frac{\theta_0+2k\pi)}{n}\right)\right], k = 0,1,2,\ldots,n-1.$$

   b). Find the cube roots of 1. Show that each cube root of 1 except 1 itself is a primitive cube root of 1.

   c). Let z be nonzero complex number and m, n be positive integer such that

   $\gcd(m,n) = d$ where $d > 1$. Prove that $(z^{\frac{1}{n}})^m = z^{\frac{\alpha}{\beta}}$ where $\alpha = \frac{m}{d}$ and $\beta = \frac{n}{d}$.

   Is $(i^{\frac{1}{4}})^6 = (i^6)^{\frac{1}{4}}$? Justify your answer.

6. a). Define the Complex Logarithmic $\log z$.

   Find all the roots of the equation $\log z = (\frac{5\pi}{2})i$.

   b). Prove or disprove each of the following statements.

   i). For each $z \in \mathbb{C} \setminus \{0\}, e^{\log z} = z$.

   ii). Let z_1, z_2 be arbitrary non-zero complex numbers. Then,

   $$\log(z_1 z_2) = \log(z_1) + \log(z_2).$$

   iii). $Log\ i^{\frac{1}{2}} = \frac{Log\ i}{2}$.

   iv). $i^{\sqrt{2}}.i^{\sqrt{3}} = i^{\sqrt{2}+\sqrt{3}}$.